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EXACT RELATIONS BETWEEN EFFECTIVE THERMOPOROELASTIC CHARACTERISTICS OF POROUS COMPOSITES

Exact relations between the effective thermoporoelastic constants of two-phase heterogeneous materials (cross-property relations) are developed by some simple way. For the composite materials with the isotropic components these relations are obtained in explicit form.

Key words: poroelastic composites, effective properties, cross-property relations

INTRODUCTION

One of the important problem of the mechanics of heterogeneous materials is the so-called homogenization problem whereby a heterogeneous material is replaced by a homogeneous one having the same response to the external loading; see the review article [9] and books [7], [10], [13]. An interesting aspect of the problem is that certain effective characteristics of the homogenized material may be interconnected (cross-property relations). The simplest cross-property relations are due to mathematical analogy between the equations that govern different phenomena: dielectric and magnetic permittivity, electric and thermal conductivity etc. In such cases, results for one of the properties imply similar results for the other one in a straightforward way. In some cases, however the cross-property relations are much less trivial. An example is provided by heat transfer in a fiber reinforced medium in direction normal to the fibers and shear deformation along the fibers (see Hashin [5], Chap. 5.5.1). A more complicated example was given by Hill [6] who derived expressions for the transversely isotropic elastic moduli of a composite material reinforced by parallel fibers. Hill [6] showed that only three of the five elastic constants are independent for such a material. Similar relations have been obtained by Schulgasser [15] for the effective elastic, piezoelectric and dielectric characteristics of transversely isotropic piezoelectric composites with continuous reinforcement.

There is another grope of relations in which the characteristics of different properties are interconnected. The relations between the effective thermal expansion coefficients and effective elastic moduli of two-phase isotropic composites were established by Levin [8] (see also Schapery, [14]) and generalized by Rosen and Hashin [12] to the case of anisotropic materials. Relations between the thermal expansion and piroelectric coefficients and electroelastic moduli of thermoelectroelastic composites were obtained by Dunn [4]. Here we suggest a simple way to obtain cross-property relations for two-phase thermoporoelastic composites. The poroelastic theory developed in the middle of the last century by Biot [2], [3] found a wide application in geomechanics for modeling of the static and dynamic behavior of real porous rocks filled with fluids. Such geological material are often highly inhomogeneous, containing the regions with different poroelastic properties (different fluids, for example, different porosity and skeleton rigidity). Thus, the obtained in this paper cross-property relations may be usefull for the solution of the homogenization problem for inhomogeneous (composite) poroelastic materials.

The linear constitutive relations for an arbitrary anisotropic poroelastic medium with incorporated thermal effects can be written in the form

$$
\begin{aligned}
\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - \alpha_{ij}p - \eta T, \\
\zeta &= \alpha_{ij}\varepsilon_{ij} + \beta p - \gamma T, \\
s &= \eta_{ij}\varepsilon_{ij} - \gamma p + cT,
\end{aligned}
\tag{1.1}
$$

where ε_{ij} and σ_{ij} are the strain and stresses, p is the pore pressure, T is the temperature change from some reference temperature, C_{ijkl} is the tensor of the frame (solid skeleton) elastic moduli, α_{ij} is the tensor of the effective stress coefficients, η_{ij} is the tensor of the thermal stress coefficients, ζ is the pore strain, s is the entropy, c is the specific heat at the constant strain, β and γ are the Biot coefficients of thermoporoelastic coupling.

EFFECTIVE THERMOPOROELASTIC PROPERTIES OF TWO-PHASE COMPOSITES

We consider now a composite poroelastic material consisting of two homogeneous generally anisotropic components (phases) occupying the regions V_1, V_2 in some representative volume element V ($V = V_1 + V_2$). Suppose that the temperature T and pore pressure p are constant in V. The constitutive relations for such a material can be written in the form

$$
\begin{aligned}
\sigma_{ij}(x) &= C_{ijkl}(x)\varepsilon_{kl}(x) - \alpha_{ij}(x)p - \eta_{ij}(x)T, \\
\zeta(x) &= \alpha_{ij}(x)\varepsilon_{ij}(x) + \beta(x)p - \gamma(x)T, \\
s(x) &= \eta_{ij}(x)\varepsilon_{ij}(x) - \gamma(x)p + c(x)T,
\end{aligned}
\tag{2.1}
$$

where $x(x_1, x_2, x_3)$ is a point in 3D-space, $C_{ijkl}(x), \alpha_{ij}(x), \eta_{ij}(x), \beta(x), \gamma(x)$ and $c(x)$ are random functions of coordinates taking the constant values $C_{ijkl}^{(1)}, \alpha_{ij}^{(1)}, ..., c^{(1)}$ when $x \in V_1$ and $C_{ijkl}^{(2)}, \alpha_{ij}^{(2)}, ..., c^{(2)}$ when $x \in V_2$.

Let us introduce the characteristic function of the first phase $V_1(x)$

($V_1(x) = 1$ when $x \in V_1$, $V_1(x) = 0$ when $x \notin V_1$). Then, the functions $C_{ijkl}(x), \alpha_{ij}(x), \eta_{ij}(x)$ can be written as

$$C_{ijkl}(x) = C_{ijkl}^{(2)} + \Delta C_{ijkl} V_1(x),$$
$$\alpha_{ij}(x) = \alpha_{ij}^{(2)} + \Delta \alpha_{ij} V_1(x), \qquad (2.2)$$
$$\eta_{ij}(x) = \eta_{ij}^{(2)} + \Delta \eta_{ij} V_1(x),$$

where

$$\Delta C_{ijkl} = C_{ijkl}^{(1)} - C_{ijkl}^{(2)}$$
$$\Delta \alpha_{ij} = \alpha_{ij}^{(1)} - \alpha_{ij}^{(2)} \qquad (2.3)$$
$$\Delta \eta_{ij} = \eta_{ij}^{(1)} - \eta_{ij}^{(2)}$$

We denote the averaging operation over the volume V as $\langle \cdot \rangle$. After averaging Eqs. (2.1) and accounting (2.2), we obtain

$$\langle \sigma_{ij}(x) \rangle = C_{ijkl}^{(2)} \langle \varepsilon_{kl}(x) \rangle + f_1 \Delta C_{ijkl} \langle \varepsilon_{kl}(x) \rangle_1 - \langle \alpha_{ij}(x) \rangle p - \langle \eta_{ij}(x) \rangle T,$$
$$\langle \zeta(x) \rangle = \alpha_{kl}^{(2)} \langle \varepsilon_{kl}(x) \rangle + f_1 \Delta \alpha_{kl} \langle \varepsilon_{kl}(x) \rangle_1 + \langle \beta(x) \rangle p - \langle \gamma(x) \rangle T, \qquad (2.4)$$
$$\langle c \rangle = \eta_{kl}^{(2)} \langle \varepsilon_{kl}(x) \rangle + f_1 \Delta \eta_{kl} \langle \varepsilon_{kl}(x) \rangle_1 + \langle \gamma(x) \rangle p - \langle c(x) \rangle T.$$

Here $\langle \cdot \rangle_1$ denotes the averaging over the region V_1, f_1 is the volume concentration of the first phase.

Due to linearity of the problem tensor $\langle \varepsilon_{ij}(x) \rangle_1$ has to be linear function of the average $\langle \varepsilon_{ij}(x) \rangle$, pore pressure p and temperature T

$$\langle \varepsilon_{ij}(x) \rangle_1 = Q_{ijkl} \langle \varepsilon_{ij}(x) \rangle + q_{ij} p + \kappa_{ij} T, \qquad (2.5)$$

where Q_{ijkl}, q_{ij} and κ_{ij} are unknown quantities that are to be determined in some homogenization process.

Substituting relation (2.5) into (2.4) we have

$$\langle \sigma_{ij}(x) \rangle = C_{ijkl}^* \langle \varepsilon_{kl}(x) \rangle - \alpha_{ij}^* p - \eta_{ij}^* T,$$
$$\langle \zeta(x) \rangle = \alpha_{kl}^* \langle \varepsilon_{kl}(x) \rangle + \beta^* p - \gamma^* T, \qquad (2.6)$$
$$\langle s(x) \rangle = \eta_{kl}^* \langle \varepsilon_{kl}(x) \rangle - \gamma^* p + c^* T,$$

where $C_{ijkl}^*, \alpha_{ij}^*, \eta_{ij}^*, \beta^*, \gamma^*$ and c^* are the effective thermoporoelastic characteristics of composite material (the quantities that connect the averaged over V fields, pore pressure and temperature). These characteristics are defined by the relations

$$C_{ijkl}^* = \langle C_{ijkl}(x) \rangle + f_1 \Delta C_{ijmn} (Q_{mnkl} - I_{mnkl}), \qquad (2.7)$$

$$\alpha_{ij}^* = \langle \alpha_{ij}(x) \rangle + f_1 \Delta \alpha_{kl} (Q_{klij} - I_{klij}) = \langle \alpha_{ij}(x) \rangle - f_1 \Delta C_{ijkl} q_{kl}, \qquad (2.8)$$
$$\eta_{ij}^* = \langle \eta_{ij}(x) \rangle + f_1 \Delta \eta_{kl} (Q_{klij} - I_{klij}) = \langle \eta_{ij}(x) \rangle - f_1 \Delta C_{ijkl} \kappa_{kl}, \qquad (2.9)$$

$$\beta^* = \langle \beta(x) \rangle + f_1 \Delta \alpha_{ij} q_{ij}, \qquad (2.10)$$
$$\gamma^* = \langle \gamma(x) \rangle - f_1 \Delta \alpha_{ij} \kappa_{ij} = \langle \gamma(x) \rangle - f_1 \Delta \eta_{ij} q_{ij}, \quad (2.11)$$
$$c^* = \langle c(x) \rangle + f_1 \Delta \eta_{ij} \kappa_{ij}, \qquad (2.12)$$

where $I_{ijkl} = \delta_{i(k} \delta_{l)j}$ and parenthesis denote symmetrization.

THE CROSS-PROPERTY RELATIONS FOR TWO-PHASE THERMOPOROELASTIC COMPOSITES

Equations (2.7) imply that

$$f_1 (Q_{ijkl} - I_{ijkl}) = (\Delta C_{ijmn})^{-1} (C_{mnkl}^* - \langle C_{mnkl}(x) \rangle). \quad (3.1)$$

Now it follows from Eqs. (2.8), (2.9)

$$\alpha_{ij}^* = \langle \alpha_{ij}(x) \rangle + \Delta \alpha_{kl} (\Delta C_{klmn})^{-1} (C_{mnij}^* - \langle C_{mnij}(x) \rangle), \quad (3.2)$$
$$\eta_{ij}^* = \langle \eta_{ij}(x) \rangle + \Delta \eta_{kl} (\Delta C_{klmn})^{-1} (C_{mnij}^* - \langle C_{mnij}(x) \rangle). \quad (3.3)$$

Because of

$$f_1 q_{ij} = -(\Delta C_{ijmn})^{-1} (\alpha_{mn}^* - \langle \alpha_{mn}(x) \rangle), \qquad (3.4)$$
$$f_1 \kappa_{ij} = -(\Delta C_{ijmn})^{-1} (\eta_{mn}^* - \langle \eta_{mn}(x) \rangle), \qquad (3.5)$$

we obtain from Eqs. (2.10) – (2.12)

$$\beta^* = \langle \beta(x) \rangle - \Delta \alpha_{ij} (\Delta C_{ijmn})^{-1} (\alpha_{mn}^* - \langle \alpha_{mn}(x) \rangle), \quad (3.6)$$

$$\gamma^* = \langle \gamma(x) \rangle + \Delta \alpha_{ij} (\Delta C_{ijmn})^{-1} (\eta_{mn}^* - \langle \eta_{mn}(x) \rangle) =$$
$$= \langle \gamma(x) \rangle + \Delta \eta_{ij} (\Delta C_{ijmn})^{-1} (\alpha_{mn}^* - \langle \alpha_{mn}(x) \rangle), \qquad (3.7)$$

$$c^* = \langle c(x) \rangle + \Delta \eta_{ij} (\Delta C_{ijmn})^{-1} (\eta_{mn}^* - \langle \eta_{mn}(x) \rangle). \quad (3.8)$$

It follows that the tensor of the effective stress coefficients α_{ij}^*, tensor of thermal stress coefficients η_{ij}^*, the Biot coefficients of thermoporoelastic coupling β^*, γ^* and specific heat c^* are not the independent characteristics, but are expressed in terms of those of two phases, the frame elastic moduli of two phases and effective elastic moduli of composite material. Equations (3.2), (3.3) and (3.6)–(3.8) are the exact results independent on the microstructure of the inhomogeneous poroelastic medium.

Hence, in any homogenization process for such a material it is necessary to find only the effective frame elastic moduli C_{ijkl}^*. Then, the other effective constants of thermoporoelastic composite material can be calculated by the formulas (3.2)–(3.8).

Suppose that the both components of the inhomogeneous poroelastic medium are isotropic with bulk frame moduli K_1, K_2, effective stress coefficients α_1, α_2, thermal stress coefficients η_1, η_2. If the composite is also macroscopically isotropic with the effective bulk modulus K^*, then the formulas (3.2), (3.3) and (3.6)–(3.8) are transformed to

$$\alpha_{ij}^* = \alpha^* \delta_{ij}, \quad \alpha^* = \langle \alpha(x) \rangle + \frac{\alpha_1 - \alpha_2}{K_1 - K_2} \left(K^* - \langle K(x) \rangle \right), \quad (3.9)$$

$$\eta_{ij}^* = \eta^* \delta_{ij}, \quad \eta^* = \langle \eta(x) \rangle + \frac{\eta_1 - \eta_2}{K_1 - K_2} \left(K^* - \langle K(x) \rangle \right), \quad (3.10)$$

$$\beta^* = \langle \beta(x) \rangle - \frac{\alpha_1 - \alpha_2}{K_1 - K_2} \left(\alpha^* - \langle \alpha(x) \rangle \right) =$$
$$= \langle \beta(x) \rangle - \left(\frac{\alpha_1 - \alpha_2}{K_1 - K_2} \right)^2 \left(K^* - \langle K(x) \rangle \right), \quad (3.11)$$

$$\gamma^* = \langle \gamma(x) \rangle = -\frac{\alpha_1 - \alpha_2}{K_1 - K_2} \left(\eta^* - \langle \eta(x) \rangle \right) =$$
$$= -\frac{(\alpha_1 - \alpha_2)(\eta_1 - \eta_2)}{(K_1 - K_2)^2} \left(K^* - \langle K(x) \rangle \right), \quad (3.12)$$

$$c^* = \langle c(x) \rangle - \frac{\eta_1 - \eta_2}{K_1 - K_2} \left(\eta^* - \langle \eta(x) \rangle \right) =$$
$$= \langle c(x) \rangle - \left(\frac{\eta_1 - \eta_2}{K_1 - K_2} \right)^2 \left(K^* - \langle K(x) \rangle \right). \quad (3.13)$$

These relations coincide with those obtained in [1], [11] by some other way.

CONCLUSIONS

The exact relations between the effective thermoporoelastic characteristics (cross-property relations) of two-phase poroelastic composites were obtained and analyzed. These relations may be used in any homogenization procedure that establish the connections between the mathematical expectations of the field variables, pore pressure and temperature in the inhomogeneous poroelastic composite material.

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