UDC 517.444

V. V. Volchkov, Vit. V. Volchkov

A NEW CHARACTERIZATION OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISK

Abstract. We study the conditions under which a function satisfying a weighted Morera property for all hyperbolic circles of a fixed radius is holomorphic. We show that one of such conditions is the restriction on a speed of decrease of the difference between the function and its Cauchy type integral.

Key words: *Cauchy integral formula, holomorphy tests, Legendre functions, hyperbolic plane*

2010 Mathematical Subject Classification: *43A45*

1. Introduction and the statement of the main result. Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. By the classical Cauchy theorem, the necessary and sufficient condition of a function $f \in C(\overline{\mathbb{D}})$ to be holomorphic in $\mathbb{D}$ is

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in \mathbb{D}. \tag{1}$$

Other holomorphy tests which are based on the Cauchi integral formula can be found in [3]. Another characterization of holomorphic functions is related with the well-known Morera property which has been studied in many contexts and generality (we refer the reader to [1], [7]–[9], [11], [12] for an account of considerable amount of research).

Throughout what follows G is the conformal automorphism group of the disk $\mathbb{D}$. We denote by gA the image of a set $A \subset \mathbb{D}$ under the map $g \in G$. For $\varrho \in (0,1)$ we set $\gamma_\varrho = \{z \in \mathbb{C} : |z| = \varrho\}$.

Let $f \in C(\mathbb{D})$ and let $E \subset (0,1)$ be a given set. Assume that

$$\int_{\gamma_\varrho} f(gz) \, dz = 0 \tag{2}$$

for all $g \in G$, $\varrho \in E$. For which E does this imply that f is holomorphic? One of the results in [2] states that f is holomorphic if and only if the equations

$$P_z^{-1}\left(\frac{1+\varrho^2}{1-\varrho^2}\right) = 0 \quad (\varrho \in E)$$

have no common solution $z \in \mathbb{C}$. Here and below P_z^{ν} is the associated Legendre function of the first kind (see [4, Ch. 3, Sect. 3.2]). Also, there is a nonconstant radial real analytic (so nonholomorphic) function on $\mathbb{D}$ satisfying (2) for one fixed $\varrho \in (0,1)$ and all $g \in G$.

In this paper we present a new aspect: we study the case when an assumption of type (1) is replaced by an upper bound of the difference

$$f(z) - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\zeta)}{\zeta - z}d\zeta$$

as $|z| \to 1$, and assumption (2) holds for one fixed $\varrho \in (0,1)$ and all $g \in G$.

Our main result is as follows.

Theorem 1. *Let $\varrho \in (0,1)$ be fixed. Then*
  (i) If $f \in C(\overline{\mathbb{D}})$ satisfies (2) for all $g \in G$ and

$$f(z) - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\zeta)}{\zeta - z}d\zeta = o\left(\sqrt{1-|z|}\right) \quad \text{as } |z| \to 1, \qquad (3)$$

then f is holomorphic on $\mathbb{D}$.
  (ii) There exists a nonholomorphic function $f \in C(\overline{\mathbb{D}})$ such that condition (2) is fulfilled for all $g \in G$, and

$$f(z) - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\zeta)}{\zeta - z}d\zeta = O\left(\sqrt{1-|z|}\right) \quad \text{as } |z| \to 1. \qquad (4)$$

The proof of Theorem 1 is based on the development of the method proposed by the authors in [10]. We introduce a transmutation operator which establishes a homeomorphism between the space of smooth radial functions in $\mathbb{D}$ and the space of even functions in $C^{\infty}(\mathbb{R})$. In a certain general sense it commutes with the generalized convolution operator; this allows us to reduce the problem to the one-dimensional case. Finally,

we use some results of the theory of convolution equations in $\mathbb{R}$ (see [7, Part 3]).

2. Notation. It is known that for each $g \in G$ there exist uniquely defined parameters $\tau, z \in \mathbb{C}$ such that $|\tau| = 1$, $|z| < 1$, and

$$gw = \tau \, \frac{w - z}{1 - \overline{z}w}$$

for each $w \in \mathbb{D}$. The group G parametrized by pairs (τ, z) is the motion group in the Poincaré model of the hyperbolic plane $\mathbb{H}^2$ which is realized as the disk $\mathbb{D}$ and has the corresponding Riemannian structure (see [5, Introduction, § 4]). The hyperbolic distance d between the points $z_1, z_2 \in \mathbb{H}^2$ in this model is defined by

$$d(z_1, z_2) = \frac{1}{2} \ln \frac{|1 - \overline{z}_1 z_2| + |z_2 - z_1|}{|1 - \overline{z}_1 z_2| - |z_2 - z_1|}.$$

In particular,

$$d(z, 0) = \frac{1}{2} \ln \frac{1 + |z|}{1 - |z|} = \operatorname{arth} |z| \quad \text{and} \quad |z| = \operatorname{th} d(z, 0), \quad z \in \mathbb{H}^2.$$

The hyperbolic measure $d\mu$ on $\mathbb{H}^2$ has the form

$$d\mu(z) = \frac{i}{2} \frac{dz \wedge \overline{dz}}{(1 - |z|^2)^2}.$$

The distance d and the measure $d\mu$ are invariant with respect to the group G.

For $r > 0$, the symbol B_r denotes the open hyperbolic disk of radius r centered at the origin of $\mathbb{H}^2$, i.e.,

$$B_r = \{z \in \mathbb{H}^2 : d(0, z) < r\}.$$

Let $B_\infty = \mathbb{H}^2$. For $r \geq 0$, we set

$$\overline{B}_r = \{z \in \mathbb{H}^2 : d(0, z) \leq r\}, \quad \partial B_r = \{z \in \mathbb{H}^2 : d(0, z) = r\}.$$

Furthermore, let χ_r be the characteristic function (the indicator) of the disk B_r. We denote by $L(\mathbb{H}^2)$ and $L_{\mathrm{loc}}(\mathbb{H}^2)$ the classes of functions integrable and locally integrable on $\mathbb{H}^2$ with respect to the measure $d\mu$,

respectively. Let dg be the Haar measure on G normalized so that the relation

$$\int\limits_{G} \psi(g0)dg = \int\limits_{\mathbb{H}^2} \psi(z)\,d\mu(z) \tag{5}$$

is valid for each function $\psi \in L(\mathbb{H}^2)$ (see [5, Introduction, § 4, Section 3]).

Let $\mathcal{D}(\mathbb{H}^2)$ (or $\mathcal{D}(\mathbb{R})$) be the set of functions with compact supports in $C^\infty(\mathbb{H}^2)$ (in $C^\infty(\mathbb{R})$, respectively) endowed with the standard topology (for instance, see [5, Ch. 2, § 2.2]). We denote the spaces of radial functions in $L(\mathbb{H}^2)$, $C^\infty(\mathbb{H}^2)$ and $\mathcal{D}(\mathbb{H}^2)$ with the induced topology by $L_\natural(\mathbb{H}^2)$, $C_\natural^\infty(\mathbb{H}^2)$ and $\mathcal{D}_\natural(\mathbb{H}^2)$. In a similar way, we let $C_\natural^\infty(\mathbb{R})$ and $\mathcal{D}_\natural(\mathbb{R})$ denote the spaces of even functions in $C^\infty(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$, respectively.

Let f_1, f_2 be radial functions in the class $L_{\mathrm{loc}}(\mathbb{H}^2)$. Assume that at least one of the functions f_1 and f_2 has compact support. Then we define the generalized hyperbolic convolution $f_1 \diamond f_2$ by

$$(f_1 \diamond f_2)(g0) = \int\limits_{\mathbb{H}^2} f_1(z) f_2(g^{-1}z)\, \frac{(1 - |z|^2)^2}{(1 - z \cdot \overline{g0})^2}\, d\mu(z), \quad g \in G. \tag{6}$$

Equality (6) shows that $f_1 \diamond f_2$ is a radial function in the class $L_{\mathrm{loc}}(\mathbb{H}^2)$.

It follows from (6) and (5) that

$$(f_1 \diamond f_2)(\zeta) = \int\limits_{G} f_1(g0) f_2(g^{-1}\zeta)\, \frac{(1 - |g0|^2)^2}{(1 - \overline{\zeta} \cdot g0)^2}\, dg, \quad \zeta \in \mathbb{D}, \tag{7}$$

and

$$f_1 \diamond f_2 = f_2 \diamond f_1.$$

If $f_1, f_2 \in C_\natural^\infty(\mathbb{H}^2)$ then, in view of (7),

$$\mathfrak{L}(f_1 \diamond f_2) = f_1 \diamond \mathfrak{L} f_2 = (\mathfrak{L} f_1) \diamond f_2, \tag{8}$$

where

$$\mathfrak{L} = 4(1 - |z|^2)^2\, \frac{\partial^2}{\partial z \partial \overline{z}} - 8(1 - |z|^2)\, z\, \frac{\partial}{\partial z}. \tag{9}$$

In addition, if $f_1, f_2, f_3 \in L_{\mathrm{loc}}(\mathbb{H}^2)$ are radial functions, and at least two of the functions f_1, f_2, f_3 have compact supports, then

$$(f_1 \diamond f_2) \diamond f_3 = f_1 \diamond (f_2 \diamond f_3).$$

Let $h \in L(\mathbb{R})$. Its Fourier transform $\widehat{h}$ is defined by

$$\widehat{h}(\lambda) = \int\limits_{-\infty}^{+\infty} h(t)e^{-i\lambda t}dt, \quad \lambda \in \mathbb{R}.$$

Assume that $\lambda \in \mathbb{C}$ and

$$\nu = \nu(\lambda) = \frac{i\lambda - 1}{2}.$$

Now define

$$U_\lambda(z) = (1 - |z|^2)^{\nu+1} F\left(\nu + 2, \, \nu + 1; \, 2; \, |z|^2\right), \quad z \in \mathbb{D}, \qquad (10)$$

where F is the Gauss hyperheometric function.

Also let

$$\mathcal{H}_\lambda(z) = (1 - |z|^2)^{\nu} F\left(\nu + 2, \, \nu; \, 1; \, |z|^2\right), \quad \lambda \in \mathbb{C}, \, z \in \mathbb{D}. \qquad (11)$$

We can conclude from (11) and the expansion of F in a hypergeometric series that $\mathcal{H}_\lambda$ is an entire function of λ. It is even because

$$\mathcal{H}_\lambda(z) = (1 - |z|^2)^{-2} F\left(\frac{3 + i\lambda}{2}, \, \frac{3 - i\lambda}{2}; \, 1; \, \frac{|z|^2}{|z|^2 - 1}\right) \qquad (12)$$

(see [4, formula 2.9 (3)]) and $F(a, b; c; z)$ is symmetric in a and b. It follows from (9), (11) and the hypergeometric differential equation that

$$(\mathfrak{L}\mathcal{H}_\lambda)(z) = -(\lambda^2 + 1)\,\mathcal{H}_\lambda(z). \qquad (13)$$

Suppose that the function $T \in L_\natural(\mathbb{H}^2)$ is compactly supported. For $\lambda \in \mathbb{C}$, let

$$\mathcal{F}(T)(\lambda) = \int\limits_{\mathbb{H}^2} T(z)\,\mathcal{H}_\lambda(z)\,(1 - |z|^2)^2\,d\mu(z).$$

Equality (12) shows that the function $\mathcal{F}(T)$ is an even entire function of λ. If f_1 and f_2 are functions with compact support in $L_\natural(\mathbb{H}^2)$ then

$$\mathcal{F}(f_1 \diamond f_2) = \mathcal{F}(f_1)\,\mathcal{F}(f_2) \qquad (14)$$

(see (8), (13) and the proof of Lemma 8 in [10]).

3. Auxiliary results. We need the following lemmas in the proof of Theorem 1.

Lemma 1. *The following equality holds*

$$\frac{\partial}{\partial \bar{z}} \left(\bar{z} \, U_\lambda(z) \right) = 2 \, \mathcal{H}_\lambda(z), \quad \lambda \in \mathbb{C}, \, z \in \mathbb{D}, \tag{15}$$

where the function U_λ is defined by (10).

Proof. First assume that $z \neq 0$. Relation (10) yields

$$\frac{\partial}{\partial \bar{z}} \left(\bar{z} \, U_\lambda(z) \right) = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho^2 (1 - \rho^2)^{\nu+1} F(\nu + 2, \, \nu + 1; \, 2; \, \rho^2) \right).$$

Using now [4, formula 2.8 (25)] and (11) we arrive at (15). Now continuous extension to the point $z = 0$ completes the proof. $\square$

For $\alpha, \beta, \lambda \in \mathbb{C}$, we set

$$\varphi_\lambda^{(\alpha,\beta)}(r) = F \left(\frac{\alpha + \beta + 1 - i\lambda}{2}, \, \frac{\alpha + \beta + 1 + i\lambda}{2}; \, \alpha + 1; \, -\text{sh}^2 r \right).$$

Lemma 2. *For each $\lambda \in \mathbb{C}$,*

$$\mathcal{F}\left(\chi_r \right)(\lambda) = \pi \, \text{sh}^2 r \, \varphi_\lambda^{(1,1)}(r). \tag{16}$$

Proof. For brevity we set

$$z = -\text{sh}^2 r, \qquad a = \frac{1 + i\lambda}{2}, \qquad b = \frac{1 - i\lambda}{2}.$$

The expansion of $\varphi_\lambda^{(\alpha,\beta)}(r)$ in a hypergeometric series shows that

$$\lim_{\alpha \to -1} \frac{\varphi_\lambda^{(\alpha,1)}(r) - 1}{\Gamma(\alpha + 1)} = \sum_{n=1}^{\infty} \frac{a(a+1)\ldots(a+n-1) \, b(b+1)\ldots(b+n-1)}{n! \, \Gamma(n)} z^n =$$

$$= z \sum_{n=0}^{\infty} \frac{a(a+1)\ldots(a+n) \, b(b+1)\ldots(b+n)}{n! \, \Gamma(n+2)} z^n = abz F(a+1, b+1; 2; z) =$$

$$= -\frac{1}{4}(\lambda^2 + 1) \, \text{sh}^2 r \, \varphi_\lambda^{(1,1)}(r).$$

Now, by the definition of the transform $\mathcal{F}$ and [8, Proposition 7.2 (ii)] we see that

$$\mathcal{F}\left(\chi_r\right)(\lambda) = \pi \int\limits_0^r \operatorname{sh}(2t)\,\varphi_\lambda^{(0,2)}(t)\,dt = -\frac{4\pi}{\lambda^2+1}\,\lim_{\alpha\to-1}\frac{\varphi_\lambda^{(\alpha,1)}(r)-1}{\Gamma(\alpha+1)}.$$

This gives, by the equality above, the desired result. $\square$

Lemma 3. *For each $r > 0$, the following assertions hold.*

(i) $\mathcal{F}\left(\chi_r\right)(0) \neq 0$.

(ii) *The function $\mathcal{F}\left(\chi_r\right)(\lambda)$ has infinitely many zeros, all of which are real, simple and lie symmetrically relative to $\lambda = 0$.*

(iii) *If $\mathcal{F}\left(\chi_r\right)(\lambda) = 0$ then $|\mathcal{F}(\chi_r)'(\lambda)| > c|\lambda|^{-3/2}$, where the constant $c > 0$ is independent of λ.*

Proof. Using (16) and [4, formula 3.2 (7)] we obtain

$$\mathcal{F}\left(\chi_r\right) = \pi \operatorname{th} r P_{(i\lambda-1)/2}^{-1}(\operatorname{ch} 2r).$$

Now the assertions of Lemma 3 are partial cases of Lemmas 2.4, 2.5 and Corollary 2.2 in [7, Part 2, Ch. 2]. $\square$

The next statements are analogs of the Paley-Wiener theorem and the inversion formula for the transform $\mathcal{F}$.

Lemma 4.

(i) *An even entire function w is the $\mathcal{F}$-transform of a function in $\mathcal{D}_\natural(\mathbb{H}^2)$ with support in $\overline{B}_r$ if and only if for each $N \in \mathbb{Z}_+$ there exists a positive constant $c_N > 0$ such that*

$$|w(\lambda)| \leq c_N\,\frac{e^{r|\operatorname{Im}\lambda|}}{(1+|\lambda|)^N}, \quad \lambda \in \mathbb{C}.$$

(ii) *Assume that $f \in (L_\natural \cap C)(\mathbb{H}^2)$ and*

$$\int\limits_0^\infty \lambda\,|\mathcal{F}(f)(\lambda)|\,d\lambda < +\infty.$$

Then

$$f(z) = \frac{16}{\pi^2}\int\limits_0^\infty \mathcal{F}(f)(\lambda)\,\mathcal{H}_\lambda(z)\,|c(\lambda)|^{-2}d\lambda + \frac{1}{2\pi}\,\mathcal{F}(f)(i)\,\mathcal{H}_i(z),$$

where

$$c(\lambda) = \frac{2^{3-i\lambda}\Gamma(i\lambda)}{\Gamma\left(\frac{i\lambda-1}{2}\right)\Gamma\left(\frac{i\lambda+3}{2}\right)}$$

and the integral is absolutely convergent for each $z \in \mathbb{C}$.

Proof. To prove (i), let $0 < \xi < t$. We set

$$K(t,\xi) = \frac{2^{3/2}}{\pi}(\operatorname{ch} t)^2(\operatorname{ch} 2t - \operatorname{ch} 2\xi)^{-1/2}F\left(2, -2; \frac{1}{2}; \frac{\operatorname{ch} t - \operatorname{ch} \xi}{2\operatorname{ch} t}\right).$$

For $\lambda \in \mathbb{C}$, $z \in \mathbb{D}$, we have

$$\mathcal{H}_\lambda(z) = \int_0^t K(t,\xi)\cos\lambda\xi d\xi,$$

where $t = \operatorname{arth}|z|$ (see [8, Proposition 7.3]). Owing to (13), the rest of the proof is identical to that of Proposition 1 in [10].

In view of Theorem 2.3 in [6], assertion (ii) can be proved in the same way as Proposition 2 in [10]. $\square$

For $f \in \mathcal{D}_\natural(\mathbb{H}^2)$, $t \in \mathbb{R}^1$, let

$$\mathfrak{A}(f)(t) = \frac{16}{\pi^2}\int_0^\infty \mathcal{F}(f)(\lambda)|c(\lambda)|^{-2}\cos(\lambda t)d\lambda + \frac{1}{2\pi}\mathcal{F}(f)(i)\cos(it).$$

Using Lemma 4 and Stirling's formula, it is easy to see that $\mathfrak{A}(f) \in C_\natural^\infty(\mathbb{R}^1)$.

Suppose that $T \in L_\natural(\mathbb{H}^2)$ has compact support. If $\mathcal{F}(T) \in L^2(\mathbb{R}^1)$ then, by the classical Paley-Wiener theorem, there exists a function $\Lambda(T) \in L^2(\mathbb{R}^1)$ with compact support such that $\widehat{\Lambda(T)} = \mathcal{F}(T)$.

The main properties of the map $f \to \mathfrak{A}(f)$ are contained in the following lemma.

Lemma 5.

(*i*) *The transformation* $\mathfrak{A}$ *can be extended to a linear homeomorphism of the spaces* $C_\natural^\infty(\mathbb{H}^2)$ *and* $C_\natural^\infty(\mathbb{R}^1)$.

(*ii*) *Let* $f \in C_\natural^\infty(\mathbb{H}^2)$, $r > 0$. *Then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}(f) = 0$ *in* $(-r, r)$.

(iii) *Assume that* $T \in L_\natural(\mathbb{H}^2)$ *has compact support and let* $\mathcal{F}(T) \in$ $\in L^2(\mathbb{R}^1)$. *Then*

$$\mathfrak{A}(f \diamond T)(t) = \mathfrak{A}(f) * \Lambda(T) \tag{17}$$

for each $f \in C_\natural^\infty(\mathbb{H}^2)$.

(iv) *Let* $\lambda \in \mathbb{C}$. *Then*

$$\mathfrak{A}(\mathcal{H}_\lambda)(t) = \cos \lambda t. \tag{18}$$

Proof. This lemma can be proved in the same way as Theorems 2 and 3 in [10] taking into account Lemma 4 and (14). $\square$

4. Proof of the main result. We now proceed to the proof of Theorem 1. To prove (i), we remark that we may confine our attention to the case $f \in C(\overline{\mathbb{D}}) \cap C^\infty(\mathbb{D})$ (otherwise we can use the standard hyperbolic regularization, see [5, Ch. 1, the proof of Theorem 4.2]). Let $r > 0$ and

$$J_f(\zeta) = \frac{1}{1 - |\zeta|^2} \int_{\partial B_r} f\left(\frac{z + \zeta}{1 + \overline{\zeta}\, z}\right)\, dz, \quad \zeta \in \mathbb{D}. \tag{19}$$

We set

$$g_\zeta z = \frac{z + \zeta}{1 + \overline{\zeta}\, z}, \quad z \in \mathbb{D}.$$

By Green's formula, we have

$$J_f(\zeta) = -\int_{B_r} \frac{\partial}{\partial \overline{z}}(f(g_\zeta z)) \frac{dz \wedge d\overline{z}}{1 - |\zeta|^2} = 2i \int_{B_r} \frac{\partial f}{\partial \overline{z}}(g_\zeta z) \frac{(1 - |z|^2)^2}{(1 + \zeta\, \overline{z})^2}\, d\mu(z).$$

Since $g_\zeta \in G$ and the measure $d\mu(z)$ is G-invariant,

$$J_f(\zeta) = 2i \int_{g_\zeta B_r} \frac{\partial f}{\partial \overline{z}}(w) \frac{(1 - |g_\zeta^{-1} w|^2)^2}{(1 + \zeta\, \overline{g_\zeta^{-1} w})^2}\, d\mu(w).$$

This implies easily that

$$J_f(\zeta) = 2i \int_{g_\zeta B_r} \frac{\partial f}{\partial \overline{z}}(w) \frac{(1 - |w|^2)^2}{(1 - w\overline{\zeta})^2}\, d\mu(w).$$

Because of (6) we can write

$$J_f = 2i \, \frac{\partial f}{\partial \bar{z}} \diamond \chi_r \quad \text{in} \quad \mathbb{H}^2. \tag{20}$$

Next, let $g \in G$, $z \in \mathbb{D}$, and

$$gz = \frac{az + b}{\bar{b}z + \bar{a}}, \quad \text{where} \quad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1. \tag{21}$$

Putting $\zeta = g0$, we see from (19) and (31) that

$$J_f(g0) = |a|^2 \int\limits_{\partial B_r} f\left(\frac{z + \frac{b}{\bar{a}}}{1 + \frac{\bar{b}}{a}z}\right) dz.$$

The change of variable $z = \dfrac{a}{\bar{a}} w$ in the integral enables us to write

$$J_f(g0) = a^2 \int\limits_{\partial B_r} f(gz) \, dz \quad \forall g \in G. \tag{22}$$

Equalities (20) and (22) show that the function f satisfies (2) for all $g \in G$ if and only if

$$\frac{\partial f}{\partial \bar{z}} \diamond \chi_r = 0 \quad \text{in} \quad \mathbb{H}^2, \tag{23}$$

where $r = \operatorname{arth} \rho$. Introduce the following auxiliary function

$$u(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} f(\zeta z) \, d\zeta, \quad z \in \mathbb{D}. \tag{24}$$

It follows from (24) that $u \in C^\infty(\mathbb{D})$. We then find that u satisfies (2) and, if $z \neq 0$,

$$u(z) = v(|z|)e^{-i \arg z}, \quad \text{where} \quad v(|z|) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(|z|e^{i\varphi}) \, e^{i\varphi} \, d\varphi. \tag{25}$$

A straightforward calculation shows that

$$\frac{\partial u}{\partial \bar{z}} = v'(|z|) + \frac{v(|z|)}{|z|}. \tag{26}$$

In particular, $\dfrac{\partial u}{\partial \overline{z}} \in C_\natural^\infty(\mathbb{H}^2)$. In addition, it follows from (25) that

$$v(0) = 0. \tag{27}$$

Relation (23) leads to

$$\frac{\partial u}{\partial \overline{z}} \diamond \chi_r = 0 \quad \text{in} \quad \mathbb{H}^2, \quad \text{where} \quad r = \operatorname{arth} \rho. \tag{28}$$

Then we find from (16) and the asymptotic expansion for $\varphi_\lambda^{(1,1)}(r)$ (see [7, Part 2, Corollary 2.2]) that $\mathcal{F}(\chi_r) \in L^2(\mathbb{R}^1)$. By the Paley-Wiener theorem, there exists a function $\Lambda_r \in L^2(\mathbb{R}^1)$ with compact support such that $\widehat{\Lambda}_r = \mathcal{F}(\chi_r)$. Owing to Lemma 5 and (17),

$$\mathfrak{A}\left(\frac{\partial u}{\partial \overline{z}}\right) * \Lambda_r = 0 \quad \text{on} \quad \mathbb{R}.$$

Using now Lemma 3 and [7, Part 3, Theorem 1.3], we conclude that

$$\mathfrak{A}\left(\frac{\partial u}{\partial \overline{z}}\right)(t) = \sum_{\lambda \in N(r)} c_\lambda \cos\left(\lambda t\right), \quad t \in \mathbb{R}, \tag{29}$$

where $N(r) = \{\lambda > 0 : \mathcal{F}(\chi_r) = 0\}$, $c_\lambda \in \mathbb{C}$, the series converges in the space $C^\infty(\mathbb{R})$, and

$$|c_\lambda| = O\left((1+\lambda)^{-\alpha}\right) \quad \text{as} \quad \lambda \to +\infty$$

for each fixed $\alpha > 0$. According to Lemma 5 the series

$$\sum_{\lambda \in N(r)} c_\lambda \, \mathcal{H}_\lambda$$

converges in $C^\infty(\mathbb{H}^2)$ to some function $w \in C_\natural^\infty(\mathbb{H}^2)$. Relations (29) and (18) yield

$$\mathfrak{A}\left(\frac{\partial u}{\partial \overline{z}}\right) = \mathfrak{A}(w),$$

whence $\dfrac{\partial u}{\partial \overline{z}} = w$ in $\mathbb{D}$. By the definition of w this shows that

$$v(|z|) = \frac{|z|}{2} \sum_{\lambda \in N(r)} c_\lambda \, U_\lambda(|z|) \tag{30}$$

(see (26), (27) and Lemma 1). Next, it follows from (25) and (3) that

$$v(|z|) = o\left(\sqrt{1-|z|}\right) \quad \text{as} \quad |z| \to 1. \tag{31}$$

In addition, for each $\varepsilon > 0$, $\lambda \in N(r)$ we have

$$U_\lambda(z) = \left(a(\lambda)e^{i\lambda t} + a(-\lambda)e^{-i\lambda t}\right)e^{-t} + O\left(e^{-2t}\right), \tag{32}$$

where

$$a(\lambda) = \frac{\Gamma(i\lambda/2)}{\sqrt{\pi}\Gamma((i\lambda+3)/2)}, \tag{33}$$

$|z| = \operatorname{th} t > \varepsilon$ and the constant in the symbol O depends only on ε (see [5, Introduction, Theorem 4.15]). Applying now (31) we see from (30) and (32) that

$$\sum_{\lambda \in N(r)} c_\lambda \left(a(\lambda)e^{i\lambda t} + a(-\lambda)e^{-i\lambda t}\right) \to 0 \quad \text{as} \quad t \to +\infty.$$

Together with (33) this implies that $c_\lambda = 0$ for all $\lambda \in N(r)$ (see, for instance, [7, Part 3, Theorem 1.6]). Owing to (30) we obtain $v = 0$.

In view of (25) and (24) this means that

$$\int_{\gamma_R} f(z)\,dz = 0$$

for each $R \in (0,1)$.

Assume now that $h \in G$. Writing (2) with $f(hz)$ instead of f and using (3) with hz instead of z, we obtain

$$\int_{\gamma_R} f(hz)\,dz = 0$$

for all $R \in (0,1)$. Then

$$\frac{\partial f}{\partial \bar{z}} \diamond \chi_r = 0 \quad \text{in} \quad \mathbb{H}^2 \quad \text{for each} \quad r > 0.$$

Now, from the arbitrariness of $r > 0$ it follows that f is holomorphic.

To prove (ii) consider the function $f(z) = \bar{z}\,U_\lambda(z)$, where $\lambda \in N(r)$ and $r = \operatorname{arth}\rho$. Owing to Lemma 1, we infer that

$$\frac{\partial f}{\partial \overline{z}} = 2\,\mathcal{H}_\lambda,$$

whence f is nonholomorphic in $\mathbb{D}$. Next, the proof of (i) shows that f satisfies (28) and (2) for all $g \in G$. Finally, it follows from [5, Introduction, Theorem 4.15] that relation (4) is fulfilled. This completes the proof of Theorem 1. $\square$

References

[1] Berenstein C. A., Chang D. C., Pascuas D., Zalcman L. *Variations on the theorem of Morera.* Contemp. Math., 1992, vol. 137, pp. 63–78.

[2] Berenstein C. A., Pascuas D. *Morera and mean-value type theorems in the hyperbolic disk.* Israel J. Math., 1994, vol. 86, pp. 61–106.

[3] Cascante C., Pascuas D. *Holomorphy tests based on Cauchy's integral formula.* Pacific J. Math., 1995, vol. 171, no. 1, pp. 89–116.

[4] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. *Higher transcendental functions.* New-York: McGraw-Hill, 1953, vols. 1, 2.

[5] Helgason S. *Groups and geometric analysis.* Pure Appl. Math., vol. 113, Orlando, FL: Academic Press, 1984.

[6] Koornwinder T. H. *Jacobi functions and analysis on noncompact semisimple Lie groups.* Special Functions: Group Theoretical Aspects and Applications (R.A. Askey et al. (eds.)). Dordrecht: D. Reidel Publishing Company, 1984, pp. 1–85.

[7] Volchkov V. V. *Integral Geometry and Convolution Equations.* Dordrecht: Kluwer Academic Publishers, 2003, 454 p.

[8] Volchkov V. V., Volchkov Vit. V. *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group.* London: Springer-Verlag, 2009, 671 p.

[9] Volchkov V. V., Volchkov Vit. V. *Offbeat Integral Geometry on Symmetric Spaces.* Basel: Birkhäuser, 2013, 592 p.

[10] Volchkov V. V., Volchkov Vit. V. *Spectral analysis on the group of conformal automorphisms of the unit disc.* Sb.: Math., 2016, vol. 207, no. 7, pp. 942–969.

[11] Zalcman L. *A bibliographic survey of the Pompeiu problem.* Approximation by solutions of partial differential equations (ed. Fuglede B. et. al). Dordrecht: Kluwer, 1992, pp. 185–194.

[12] Zalcman L. *Supplementary bibliography to "A bibliographic survey of the Pompeiu problem".* Contemp. Math. Radon Transform and Tomography, 2001, vol. 278, pp. 69–74.

Valerii V. Volchkov
Donetsk National University
24, Universitetskaya str., Donetsk 83001
E-mail: valeriyvolchkov@gmail.com

Vitalii V. Volchkov
Donetsk National University
24, Universitetskaya str., Donetsk 83001
E-mail: volna936@gmail.com