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ON THE PROBLEM OF DETERMINING PARAMETERS IN THE SCHWARZ EQUATION

Abstract. P. P. Kufarev's method makes it possible to reduce the problem of determining the parameters in the Schwarz-Christoffel integral to the problem of successive solutions of systems of ordinary differential equations. B. G. Baibarin obtained a generalization of this method for the problem of determining parameters (preimages of vertices and accessory parameters) in the Schwarz differential equation, whose solution is a holomorphic univalent mapping from the upper half-plane onto a circular-arc polygon. This paper specifies the initial condition for the system of differential equations for the parameters of the Schwarz equation obtained by B. G. Baibarin. This method is used to solve the problem of determining the accessory parameters for some particular mappings.

Key words: *conformal mapping, Schwarz equation, accessory parameters, parametric method, circular-arc polygon*

2010 Mathematical Subject Classification: *30C20*

1. Introduction. There is a classical approach for constructing a conformal mapping from the canonical domain (unit disk or complex half-plane) onto a circular-arc polygon based on the Schwarz differential equation. The well-known problem of determining the parameters of this equation is solved for certain particular mappings onto circular-arc polygons. The simplest and most studied cases are when the circular-arc polygon has no more than three vertices [7], [14]. The parameter problem is solved for some more complicated particular cases on the basis of the P. Ya. Polubarinova-Kochina method in works by P. Ya. Polubarinova-Kochina, E. N. Bereslavsky and others, see for example [4] and the overview work [3]. We also mark the approach of A. R. Tsitskishvili, based on the theory of conjugation for several unknown functions and on the theory of I. A. Lappo-Danilevskii [17], [18], which allows us obtaining some particular cases. However, it is oriented to general studies, as well as the work of L. I. Chibrikova (for example [5]).

These papers are of interest both for constructing conformal mappings and for studying differential equations of the Fuchs class. The conformal mapping of a canonical domain onto a polygon with a boundary consisting from segments can be represented by the Schwarz-Christoffel integral. The problem of determining the parameters of these mapping is simpler, since for an n-polygon with a straight-line boundary it is sufficient to define n preimages of vertices, while for a circular-arc n-polygon it is necessary to define $2n$ parameters: n preimages of vertices and n additional parameters, called accessory ones. In 1947, P. P. Kufarev [15] (see also [1], [16]) proposed a method for defining preimages of vertices in the Schwarz-Christoffel integral for mapping from the unit disk onto a polygon with internal normalization. For special cases, the method was first tested in the work of Yu. V. Chistyakov [6], then in was applied in [9]. The method is convenient for practice, it received various generalizations. Thus, the method is extended to mappings with boundary normalization in the work of V. Ya. Gutlyansky and A. O. Zaidan [8]. L. Yu. Nizamayeva [12], [11] proposed a new approach of finding the parameters in the Schwarz-Christoffel integral, using the idea of P. P. Kufarev and the technique of Hilbert boundary value problems with piecewise smooth coefficients and variations of such problem solutions. In the work of N. N. Nakipov and S. R. Nasyrov [10], the method is generalized to mappings onto multisheet polygons containing branch points. In [13] Kufarev's method is extended for mappings from a half-plane onto numerable polygons with transfer symmetry. B. G. Baibarin generalizes P. P. Kufarev's method for the problem of determining parameters in the Schwarz differential equation, representing a holomorphic and univalent mapping from the upper half-plane onto a circular-arc polygon. This paper specifies the result of B. G. Baibarin [2]. With help of the generalization obtained by B. G. Baibarin we define accessory parameters for some particular mappings.

2. On Kufarev's method. We briefly describe P. P. Kufarev's method. Suppose we need to obtain a conformal map from the upper half-plane onto some circular polygon Δ. Without loss of generality, we can assume it contains the origin and that the polygon Δ is a kernel with respect to the origin of some family of simply connected domains

$\Delta(t)$, $0 \leqslant t \leqslant T$. Here the family $\Delta(t)$ is obtained by carrying out a cut along N arcs of circles in some initial domain Δ_0, $\Delta(0) = \Delta_0$, and $\Delta(T) = \Delta$. There is a family of functions $f = f(z,t)$ that maps the upper half-plane $\Pi^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ onto $\Delta(t)$. In the first step, carrying a cut along the first arc, we have a family of mappings $f = f(z,t)$, represented by Schwarz's differential equation on the one hand, and the Loewner equation on the other. Note that the parameters of the map $f = f(z,t)$ (the preimages of the vertices of the polygon $\Delta(t)$ and the accessory parameters) change continuously with the length of the cut. Using the differential equations of Loewner and Schwarz, one can obtain a system of ordinary differential equations for the parameters of the map f. In the paper of P. P. Kufarev [15], devoted to the determination of parameters in the Schwarz-Christoffel integral, the initial conditions of the ODE system at the first step are the parameters of the map $f = f(z, 0)$ (the initial domain Δ_0 can be chosen sufficiently simple to write explicitly the map $f = f(z, 0)$). In the generalization of P. P. Kufarev's method to the case of mappings onto circular-arc polygons, additional difficulties arise in determining the initial conditions of the ODE system (for more about this, see the following paragraphs). Let us integrate the ODE system for the corresponding value of the parameter $t = t_1$, that is, we define the parameters of the function $f = f(z, t_1)$ that maps the upper half-plane onto the domain $\Delta(t_1)$ (the domain Δ_0 with a cut along the first arc). Then we can proceed to the second step and carry out the cut along the second arc and define the parameters of the corresponding family of functions. Thus, in N steps, we can define the parameters of the map $f = f(z, T)$.

3. The main results of B. G. Baibarin generalization. In this section we present the main results of the work [2].

Let $L(t) = \{\zeta : \zeta = \zeta(\tau), t_1 < \tau < t\}$, $0 \leqslant t_1$, $t \leqslant t_{\frac{n}{2}+1}$, $\zeta(t_p) = \zeta_p$, $p = 1,\ldots, \frac{n}{2}$, $\zeta_1 > 0$, be a piecewise smooth curve consisting of circular arcs that does not pass through the origin (n is an even number). Denote by $\Delta(t)$ a domain, obtained from the plane by carrying out a cut along the positive part of the real axis from the point ζ_1 to infinity and excluding the curve L (the curve L and the cut intersect only at the point ζ_1), $\Delta(t) = \mathbb{C}\backslash\big(L(t) \bigcup \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \geqslant \zeta_1, \operatorname{Im}\zeta = 0\}\big)$. Let the family of functions $w = w(z,t)$, $t_1 \leqslant t \leqslant t_{\frac{n}{2}+1}$, map the half-plane onto the family $\Delta(t)$ (Fig. 1), such that $w\big(\beta(t),t\big) = 0$, where β satisfies the differential equation

$$\frac{d\beta(t)}{dt} = \frac{1}{\beta(t) - \widetilde{\lambda}(t)}, \ \beta(t_1) = \beta_1. \tag{1}$$

Here $\widetilde{\lambda}(t)$ is the preimage of the movable end of the cut $\zeta(t)$. Let the infinity be fixed under the mapping $w(z,t)$ and $w(\cdot,t)$ maps points $\widetilde{a}_{\frac{n}{2}+1-p}(t)$, $\widetilde{a}_{\frac{n}{2}+p}(t)$ in point ζ_p, $p = 1, \dots, \frac{n}{2}$. At the vertex, whose preimage is the point $\widetilde{a}_p$, we denote the inner angle of $D(t)$ by $\alpha_p\pi$. Therefore, we have $\alpha_p = 2 - \alpha_{n-p+1}$. Note that we can choose any convenient domain [1] as

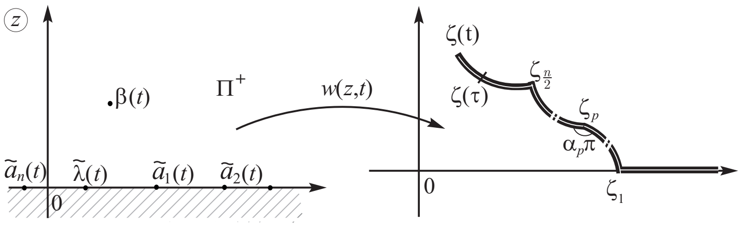


Figure 1: The family $w(z,t)$

the initial region $\Delta(0)$.

Denote the Schwarz derivative of the mapping w by S,

$$S(w,t) := \frac{w'''(z,t)}{w'(z,t)} - \frac{3}{2}\left(\frac{w''(z,t)}{w'(z,t)}\right)^2.$$

A family of mappings $w = w(z,t)$ satisfies the Schwarz differential equation

$$S(z,t) = \sum_{p=0}^{n}\left(\frac{L_p}{(z-\widetilde{a}_p(t))^2} + \frac{\widetilde{M_p}(t)}{z-\widetilde{a}_p(t)}\right), \tag{2}$$

where $\widetilde{\lambda}(t) = \widetilde{a_0}(t)$, $L_p = \dfrac{1}{2}(1-\alpha_p^2)$.

Since w satisfies the Loewner differential equation in the half-plane

$$\frac{\partial w(z,t)}{\partial t} + \frac{1}{z-\widetilde{\lambda}(t)}\frac{\partial w(z,t)}{\partial z} = 0, \quad w(z,t_1) = \zeta_1 + z^2,$$

the Schwarz derivative S satisfies the equation

$$\frac{\partial S(z,t)}{\partial t} + \frac{1}{z-\widetilde{\lambda}(t)}\frac{\partial S(z,t)}{\partial z} - \frac{2S(z,t)}{\left(z-\widetilde{\lambda}(t)\right)^2} - \frac{6}{\left(z-\widetilde{\lambda}(t)\right)^4} = 0 \tag{3}$$

with the initial condition $S(z,t_0) = -\frac{3}{2}\frac{1}{z^2}$.

The function on the left-hand side of (3) has poles of the third order at the points $z = \widetilde{a}_p$, $p = 1,\dots,n$. At the point $z = \widetilde{a}_0 = \widetilde{\lambda}$ it has a pole of the forth order. On the other hand, in the right-hand side of the equality (3) the function is identically equal to zero. It follows that the parameters $\widetilde{a}_p$, $\widetilde{M}_p$, $p = 0,1,\dots,n$, of the Schwarz derivative S of the function w satisfy the system of ordinary differential equations

$$\begin{cases} \widetilde{a}_p'(t) = \dfrac{1}{\widetilde{a}_p(t) - \widetilde{\lambda}(t)}, \quad p = 1,\dots,n, \\ \widetilde{\lambda}'(t) = -\widetilde{M}_0(t), \\ \widetilde{M}_p'(t) + L_p\left(\widetilde{a}_p'(t)\right)^3 - \left(\widetilde{a}_p'(t)\right)^2 \widetilde{M}_p(t) = 0,\ p = 1,\dots,n, \\ \displaystyle\sum_{p=0}^{n} \widetilde{M}_p'(t) = 0. \end{cases} \tag{4}$$

Next, we make the change of variable $x^2 = t - t_{\frac{n}{2}}$. Then the parameters

$$\widetilde{a}_p(t) = \widetilde{a}_p(x^2 - t_{\frac{n}{2}}) = a_p(x),\ \widetilde{M}_p(t) = \widetilde{M}_p(x^2 - t_{\frac{n}{2}}) = M_p(x),\ p = 1,\dots,n,$$

$$\widetilde{\lambda}(t) = \widetilde{\lambda}(x^2 - t_{\frac{n}{2}}) = \lambda(x), \quad \widetilde{M}_0(t) = \widetilde{M}_0(x^2 - t_{\frac{n}{2}}) = M_0(x),$$

as functions of the variable x are expanded in series

$$\begin{aligned} &a_0(x) = \lambda(x) = \sigma + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \dots, \\ &a_p(x) = a_{p0} + a_{p1}x + a_{p2}x^2 + a_{p3}x^3 + \dots, \quad p = 1,\dots,n, \\ &M_p(x) = m_{p0} + m_{p1}x + m_{p2}x^2 + m_{p3}x^3 + \dots, \quad p = 2,\dots,n-1, \\ &M_p(x) = \frac{m_{p,-1}}{x} + m_{p0} + m_{p1}x + m_{p2}x^2 + \dots, \quad p = 0,1,n, \end{aligned} \tag{5}$$

and satisfy the system of differential equations obtained from the system (4) by changing of variable $t = x^2 - t_{\frac{n}{2}}$:

$$\begin{cases} a_p'(x) = \dfrac{2x}{a_p(x) - \lambda(x)}, \quad p = 1,\dots,n, \\ \lambda'(x) = -2xM_0(x), \\ 2x^2 M_p'(x) + L_p\left(a_p'(x)\right)^3 - x\left(a_p'(x)\right)^2 M_p(x) = 0,\ p = 1,\dots,n, \\ \displaystyle\sum_{p=0}^{n} M_p'(x) = 0. \end{cases} \tag{6}$$

The values $a_{10} = a_{n0} = \sigma$, a_{p0}, m_{p0}, $p = 2, \ldots, n-1$, are known if the mapping $w = w(z, t_{\frac{n}{2}})$ is specified.

There are the following particular integrals of the system (4)

$$\sum_{p=0}^{n} M_p = 0, \tag{7}$$

$$\sum_{p=0}^{n} a_p M_p = -\frac{3}{2} - \sum_{p=0}^{n} L_p,$$

$$M_0^2 + 2\sum_{p=1}^{n} \left(\frac{L_p}{(a_p - \lambda)^2} - \frac{M_p}{(a_p - \lambda)^3} \right) = 0.$$

Substituting the expansions (5) into the system (6), for the coefficients λ_1, a_{p1}, $p = 1, \ldots, n$, $m_{0,-1}$, $m_{1,-1}$, $m_{n,-1}$, $m_{p,1}$, $p = 2, \ldots, n-1$, we obtain a system of algebraic equations that has a real solution ($\alpha_1, \alpha_n \neq 0, 2$)

$$a_{11} = \sqrt{2\frac{\alpha_n}{\alpha_1}}, \quad a_{n1} = -\sqrt{2\frac{\alpha_1}{\alpha_n}}, \quad \lambda_1 = a_{11} + a_{n1},$$

$$m_{0,-1} = -\frac{\lambda_1}{2}, \quad m_{1,-1} = L_1 \frac{a_{11}^3}{a_{11}^2 + 2}, \quad m_{n,-1} = L_n \frac{a_{n1}^3}{a_{n1}^2 + 2}, \tag{8}$$

$$a_{p1} = 0, \quad p = 2, \ldots, n-1, \quad m_{p1} = 0, \quad p = 2, \ldots, n-1.$$

To determine $2n+2$ second coefficients of the series (5) we get $2n+1$ linearly independent equations

$$\begin{cases} a_{12} = \lambda_2 \dfrac{a_{11}^2}{4 + a_{11}^2}, a_{n2} = \lambda_2 \dfrac{a_{n1}^2}{4 + a_{n1}^2}, a_{p2} = \dfrac{1}{a_{p0} - \sigma}, \ p = 2, \ldots, n-1, \\ m_{00} = -\lambda_2, \ m_{10} = 2L_1 a_{12} \dfrac{a_{11}^2 + 6}{a_{11}^2 + 2}, \ m_{n0} = 2L_n a_{n2} \dfrac{a_{n1}^2 + 6}{a_{n1}^2 + 2}, \\ m_{p2} = a_{p2}^2 \big(m_{p0} - 2L_p a_{p2}\big), \quad p = 2, \ldots, n-1. \end{cases} \tag{9}$$

To determine $2n+2$ of the third coefficients, we get a system of $2n+1$

linearly independent equations

$$\begin{cases} a_{13} = \dfrac{8\lambda_2^2 a_{11}^3}{(a_{11}^2+4)^2(a_{11}^2+6)} + \dfrac{a_{11}^2}{a_{11}^2+6}\lambda_3, \\ a_{n3} = \dfrac{8\lambda_2^2 a_{n1}^3}{(a_{n1}^2+4)^2(a_{n1}^2+6)} + \dfrac{a_{n1}^2}{a_{n1}^2+6}\lambda_3, \\ m_{01} = -\dfrac{3}{2}\lambda_3, \quad m_{11} = 3L_1\dfrac{a_{11}^4}{a_{11}^4-4}\lambda_3, \quad m_{n1} = 3L_n\dfrac{a_{n1}^4}{a_{n1}^4-4}\lambda_3, \\ a_{p3} = \dfrac{2}{3}\dfrac{\lambda_1}{(a_{p0}-\sigma)^2}, \quad p = 2,\dots,n-1, \\ m_{p3} = 2a_{p2}a_{p3}\big(m_{p0} - 3L_p a_{p2}\big), \quad p = 2,\dots,n-1. \end{cases} \tag{10}$$

To determine the fourth and subsequent coefficients a_{pk}, $p = 1,\dots,n$, $m_{0,k-2}$, $m_{1,k-2}$, $m_{n,k-2}$, m_{pk}, $p = 2,\dots,n-1$, substitute the expansions (5) into the system (6) to get $2n+2$ linearly independent equations, $k = 4,5,\dots$. Thus, the fourth and subsequent coefficients in the expansion (5) are determined by the series method. In [2], the convergence of the series (5) whose coefficients are found by this method is proved. Specifying the initial conditions of a system of ordinary differential equations (6) (finding the coefficients λ_2 and λ_3) requires additional effort.

4. Addition to B.G. Baibarin's results. It is not possible to find the coefficients λ_2 and λ_3 with the help of the work [2].

Substituting the series (5) into the equality (7) and equating the free terms we obtain

$$\sum_{p=0}^{n} m_{p0} = 0,$$

that is

$$\lambda_2 = \frac{(2+\alpha_1)(2+\alpha_n)}{9\alpha_1\alpha_n}\sum_{p=2}^{n-1} m_{p0}. \tag{11}$$

The coefficient λ_3 depends on the curvature of the arc, along which the cut at the current step is carried out, but the relationship between λ_3 and the curvature of the arc is not established.

Note that the expansion of the Schwarz derivative S of the function w at infinity has the form

$$S(w,z) = \frac{1-\alpha_\infty^2}{2z^2} + \frac{b_1}{z^3} + \frac{b_2}{z^4} + \dots,$$

where $\alpha_\infty \pi = -2\pi$ is angle at infinity, b_1,b_2 are real constants. So one can write [14] the equation (2) as

$$S(w,z) = \sum_{p=0}^{n} \frac{L_p}{(z-a_p)^2} + \frac{\left(2-n-\alpha_\infty^2 + \sum\limits_{j=0}^{n} \alpha_j^2\right) z^{n-1} + \sum\limits_{v=0}^{n-2} \gamma_v z^v}{2\prod\limits_{j=0}^{n}(z-a_j)},$$

here γ_v are some real parameters. Now we can turn from $n+1$ unknowns parameters $M_0, M_1, \ldots, M_n$ to $n-1$ unknown parameters $\gamma_0, \ldots, \gamma_{n-2}$ by the formula

$$M_p = \frac{\left(2-n-\alpha_\infty^2 + \sum\limits_{j=0}^{n} \alpha_j^2\right) a_k^{n-1} + \sum\limits_{v=0}^{n-2} \gamma_v a_k^v}{2\prod\limits_{j=0, j\neq k}^{n}(a_k-a_j)}, \quad p = 0, \ldots, n. \tag{12}$$

5. A particular cases. Let us consider the particular case when $n = 2$. The function $w = w(z,t)$ maps the upper half-plane onto a circular-arc polygon $\Delta(t)$, which is a plane with a cut along the ray from the point $\zeta_1 > 0$ to infinity and with a cut along the arc of the circle starting from the point ζ_1. There are four preimages of vertices under the mapping w, they are λ, a_1, a_2 and infinity, $a_2 \leqslant \lambda \leqslant a_1$, $w(\infty,t) = \infty$; the angles at the corresponding vertices are equal to 2π, $\alpha\pi$, $(2-\alpha)\pi$, -2π. We note that, according to the chosen normalization (1) $f(z,0) = z^2 + \zeta_1$. The mapping $w = w(z,t)$ satisfies the Schwarz differential equation. With the help of the formula (12), the accessory parameters of this equation can be written in the form

$$M_0 = \frac{2\lambda(1-\alpha)^2 + \gamma_0}{2(\lambda - a_1)(\lambda - a_2)},$$

$$M_1 = \frac{2a_1(1-\alpha)^2 + \gamma_0}{2(a_1 - \lambda)(a_1 - a_2)},$$

$$M_2 = \frac{2a_2(1-\alpha)^2 + \gamma_0}{2(a_2 - \lambda)(a_2 - a_1)}.$$

The ODE system (6) takes the form

$$\begin{cases} a_p'(x) = \frac{2x}{a_p(x)-\lambda(x)}, \quad p=1,2, \\ \lambda'(x) = -2xM_0(x), \\ 2x^2M_p'(x) + L_p\left(a_p'(x)\right)^3 - x\left(a_p'(x)\right)^2 M_p(x) = 0,\ p=1,2, \\ \sum\limits_{p=0}^{n} M_p'(x) = 0, \end{cases}$$

hence, taking into account that $m_{01} = -1.5\lambda_3$, we obtain

$$\gamma_0(x) = -2\big(a_1(3-\alpha) + a_2(1+\alpha) - \lambda(3-\alpha)(1+\alpha)\big).$$

The system of ODE can now be written in the form

$$\begin{cases} a_p'(\tau) = \dfrac{1}{a_p(\tau)-\lambda(\tau)}, \quad p=1,2, \\ \lambda'(\tau) = -\dfrac{2\lambda(\tau)(1-\alpha)^2 + \gamma_0(\tau)}{2\big(\lambda - a_1(\tau)\big)\big(\lambda - a_2(\tau)\big)}, \end{cases}$$

where $\tau = x^2$, or as follows

$$\begin{cases} \dfrac{da_1(\lambda)}{d\lambda} = \dfrac{\lambda - a_2(\lambda)}{4\lambda - a_1(\lambda)(3-\alpha) - a_2(\lambda)(1+\alpha)}, \\ \dfrac{da_2(\lambda)}{d\lambda} = \dfrac{\lambda - a_1(\lambda)}{4\lambda - a_1(\lambda)(3-\alpha) - a_2(\lambda)(1+\alpha)}. \end{cases}$$

After the substitution

$$p(\lambda) = (1+\alpha)\big(a_2(\lambda) - \lambda\big) + (3-\alpha)\big(a_1(\lambda) - \lambda\big),$$

$$q(\lambda) = (3-\alpha)\big(a_2(\lambda) - \lambda\big) + (1+\alpha)\big(a_1(\lambda) - \lambda\big),$$

the system becomes simpler:

$$\begin{cases} \dfrac{dp(\lambda)}{d\lambda} = \dfrac{q(\lambda)}{p(\lambda)} - 4, \\ \dfrac{dq(\lambda)}{d\lambda} = -3. \end{cases}$$

Hence, taking into account the normalization $a_1(0) = a_2(0) = \lambda(0) = 0$, conditions (8) and conditions (9), (10), (11), which in this case take the form $\lambda_2 = a_{12} = a_{22} = 0$, $a_{13} = \dfrac{2-\alpha}{2(1+\alpha)}\lambda_3$, $a_{23} = \dfrac{\alpha}{2(3-\alpha)}\lambda_3$, we obtain

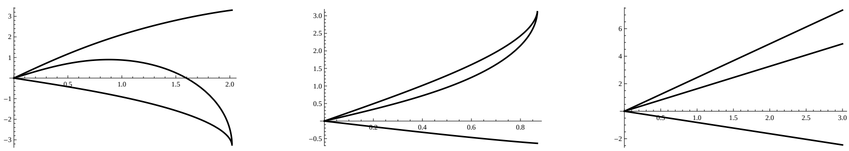
$$\lambda = a_1(1+\alpha) + a_2(3-\alpha),$$

$$(a_2 - a_1)^3 = c\big(a_2(2-\alpha) + a_1\alpha\big),$$

where

$$c = 8\sqrt{2\alpha(2-\alpha)}\,\frac{(\alpha-3)(1+\alpha)}{(2-\alpha)^3\alpha^3\lambda_3}.$$

Consider the behavior of the parameters $a_1 = a_1(x)$, $a_2 = a_2(x)$, $\lambda = \lambda(x)$, using their expansions (5), for fixed λ_3 and α.



a) $\lambda_3 = -1$ b) $\lambda_3 = 1$ c) $\lambda_3 = 0$

Figure 2: The graphs of a_1, a_2 and λ

Let $\alpha = 0.5$, $\lambda_3 = -1$. Then $x \in [0; X]$, $X \approx 2.0205$, and $\lim\limits_{x\to X} \lambda(x) = = \lim\limits_{x\to X} a_2(x) \approx -3.28$, $\lim\limits_{x\to X} a_2(x) \approx 3.28$. Graphs of functions a_1, a_2 and λ are shown in Fig. 2a) (the graph of a_1 is on the top, the graph of λ is the middle one, and the graph of a_2 is on the bottom). Hence, we conclude that this case corresponds to the mapping w from the half-plane onto the circular-arc polygon shown in Fig. 3a).

Let $\alpha = 0.5$, $\lambda_3 = 1$. Then $x \in [0; X]$, $X \approx 0.869$, and $\lim\limits_{x\to X} a_2(x) \approx -0.634$, $\lim\limits_{x\to X} \lambda(x) = \lim\limits_{x\to X} a_1(x) \approx 3{,}14$. Graphs of functions a_1, a_2, λ are shown in Fig. 2b) (the graph of a_1 is on the top, the graph of λ is the middle one, and the graph of a_2 is on the bottom). Hence we can conclude that this case corresponds to the mapping w from the half-plane onto the circular-arc polygon shown in Fig. 3b).

Let $\alpha = 0.5$, $\lambda_3 = 0$. Then a_1, a_2 and λ are linear functions of the variable x, $x \in [0; +\infty)$, their graph are shown in Fig. 2c). This case corresponds to the mapping w from the half-plane onto the polygon shown in Fig. 3c).

Next, we carry out another cut ($n = 4$) and let

$$\lambda_3 = \frac{16\sqrt{2}(\beta-1)}{81u^2(2-3k-\alpha)^2(3k+\alpha)^2\sqrt{(2-\beta)^3\beta^3}}\Big(23\beta^2 - 46\beta - 94 - \\ -\alpha^2(13 - 8\beta + 4\beta^2) + 2\alpha(13 - 8\beta + 4\beta^2) - \\ -9k^2(133 + 22\beta - 11\beta^2) + 3k(1-\alpha)(133 + 22\beta - 11\beta^2)\Big),$$

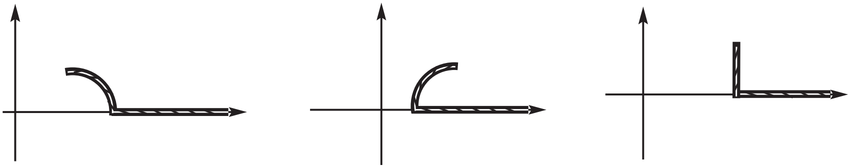

a) $\lambda_3 = -1$ b) $\lambda_3 = 1$ c) $\lambda_3 = 0$

Figure 3: The range of w

where $u = a_3(0) - a_2(0)$, $k = \dfrac{u^2}{c}$, $\beta = \alpha_1$, $\alpha = \alpha_2$. We have five preimages of the vertices $a_3(x) \leqslant a_4(x) \leqslant \lambda(x) \leqslant a_1(x) \leqslant a_2(x)$ under the mapping w. Note that $\lambda(0) = a_1(0) = a_4(0)$, $a_3(0)$, $a_2(0)$, c have the values defined in the previous step.

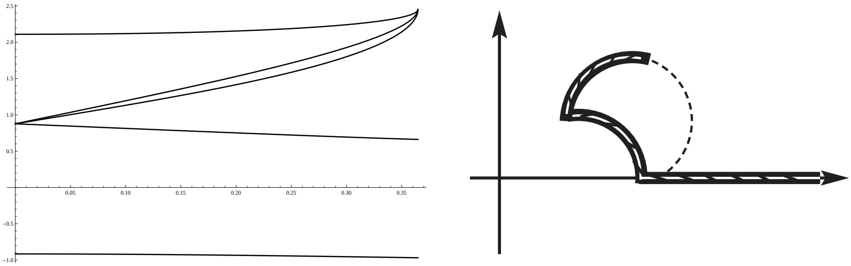
The parameters M_p, $p = 0, 1, 2, 3, 4$, can be written according to the formula (12):

$$\frac{\gamma_0}{2} = (\alpha-3)(1-\beta)\lambda a_1 a_2 - (1+\alpha)(1-\beta)\lambda a_1 a_3 + (3-\alpha)(1+\alpha)\lambda a_1 a_4 -$$
$$-(1-\beta)^2\lambda a_2 a_3 + (3-\alpha)(1-\beta)\lambda a_2 a_4 + (1+\alpha)(1-\beta)\lambda a_3 a_4 +$$
$$+(1-\beta)a_1 a_2 a_3 - (3-\alpha)a_1 a_2 a_4 - (1+\alpha)a_1 a_3 a_4 - (1-\beta)a_2 a_3 a_4,$$

$$\frac{\gamma_1}{2} = \lambda a_1\big((1-\alpha)^2 - 4\beta\big) + \lambda a_2(1-\beta)^2 + \lambda a_3(1-\beta)^2 +$$
$$+\lambda a_4\big((1-\alpha)^2 - 4(2-\beta)\big) + a_1 a_2\big(5 - 2\beta - \alpha(2-\beta)\big) +$$
$$+a_1 a_3\big(1 + \alpha(2-\beta)\big) + a_1 a_4(1-\alpha)^2 + a_2 a_3(1-\beta)^2 +$$
$$+a_2 a_4\big(1 + \beta(2-\alpha)\big) + a_3 a_4(1+\alpha\beta),$$

$$\frac{\gamma_2}{2} = \lambda\big((3-\alpha)(1+\alpha) - (1-\beta)^2\big) - a_1\big((1-\alpha)^2 + 3(1-\beta)\big) +$$
$$+a_2\big(\alpha - 4 + \beta(2-\beta)\big) + a_3\big(\beta(2-\beta) - 2 - \alpha\big) + a_4\big(3(1-\beta) - (1-\alpha)^2\big).$$

Indeed, it can be verified directly that M_p satisfies the system (6) and the conditions (8), (9), (10). Consider behavior of the parameters a_1, a_2, a_3, a_4 and λ for the chosen λ_3 and $\alpha = 0.5$, $\beta = 1/3$. Suppose that, for the first arc, $\lambda_3 = -1$ and $x = 1$. Then $\lambda(0) = a_1(0) = a_4(0) \approx 0.87689$, $a_3(0) \approx -0.91502$, $a_2(0) \approx 2.10963$. For the second arc, we have $x \in [0; X], X \approx 0.3647$, and $\lim\limits_{x\to X} \lambda(x) = \lim\limits_{x\to X} a_1(x) = \lim\limits_{x\to X} a_2(x) \approx 2.4502$, $\lim\limits_{x\to X} a_3(x) \approx -0.9677$, $\lim\limits_{x\to X} a_4(x) \approx 0.6618$. Function graphs of a_2, a_1, λ, a_4 and a_3 are shown, from top to bottom, in



a) Behavior of the parameters b) The range of w

Figure 4: Second step

Fig 4a). In this case, we see that, as x tends to X, the second arc approaches to a point of the ray, as shown in Fig. 4b).

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