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SINGULAR POINTS FOR THE SUM OF A SERIES OF EXPONENTIAL MONOMIALS

Abstract.

A problem of distribution of singular points for sums of series of exponential monomials on the boundary of its convergence domain is studied. The influence of a multiple sequence $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ of the series in the presence of singular points on the arc of the boundary, the ends of which are located at a certain distance R from each other, is investigated. In this regard, the condensation indices of the sequence and the relative multiplicity of its points are considered. It is proved that the finiteness of the condensation index and the zero relative multiplicity are necessary for the existence of singular points of the series sum on the R-arc. It is also proved that for one of the sequence classes Λ, these conditions give a criterion. Special cases of this result are the well-known results for the singular points of the sums of the Taylor and Dirichlet series, obtained by J. Hadamard, E. Fabry, G. Pólya, W.H.J. Fuchs, P. Malliavin, V. Bernstein and A. F. Leont'ev, etc.

Key words: *invariant subspace, series of exponential monomials, singular point, convex domain*

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Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and its multiplicities n_k, $|\lambda_{k+1}| \geqslant |\lambda_k|$ and $|\lambda_k| \to \infty$. We denote by $n(r, \Lambda)$ the number of points λ_k (taking into account their multiplicities) located in the disk $B(0, r)$. The upper density and maximal density of Λ are the quantities

$$\bar{n}(\Lambda) = \limsup_{r\to\infty} \frac{n(r,\Lambda)}{r}, \quad n^0(\Lambda) = \lim_{\delta\to 0} \limsup_{r\to\infty} \frac{n(r,\Lambda) - n((1-\delta)r,\Lambda)}{\delta r}.$$

We consider the series of exponential monomials

$$\sum_{k=1,n=0}^{\infty,n_k-1} a_{k,n} z^n e^{\lambda_k z}. \tag{1}$$

Let $a = \{a_{k,n}\}$ and $D(\Lambda, a)$ be an open kernel of the set of all points $z \in \mathbb{C}$ where the series (1) converges and its sum is an analytical function. We denote the sum of the series (1) by $g_{\Lambda,a}$ and the set of all sequences of coefficients $a = \{a_{k,n}\}$ for which $D(\Lambda, a) \neq \emptyset$ by $\mathcal{U}(\Lambda)$. In this article we observe the problem of distribution of singular points for the function $g_{\Lambda,a}$ on a boundary $\partial D(\Lambda,a)$.

Let $\Theta(\Lambda)$ be the set of all partial limits of the sequence $\{\bar{\lambda}_k/|\lambda_k|\}_{k=1}^{\infty}$. We assume

$$m(\Lambda) = \limsup_{k\to\infty} n_k/|\lambda_k|.$$

In [6] we showed that in the general case the set $D(\Lambda,a)$ may not be convex and is not even connected. But if $m(\Lambda) = 0$ and $\bar{n}(\Lambda) < \infty$ then the Cauchy-Hadamard theorem for series of exponential monomials (see [6], Theorem 4.1) shows that $D(\Lambda, a)$ is a convex domain:

$$D(\Lambda, a) = \{z : \mathrm{Re}(ze^{-i\Theta}) < h(\Theta, a, \Lambda), e^{i\Theta} \in \Theta(\Lambda)\}, \tag{2}$$

$$h(\varphi, a, \Lambda) = \inf \left(\liminf_{j\to\infty} \min_{0\leqslant n \leqslant n_{k(j)}-1} \frac{\ln(1/|a_{k(j),n}|)}{|\lambda_{k(j)}|} \right), \tag{3}$$

where the infimum is taken with respect to all subsequences $\{\lambda_{k(j)}\}$ such that $\lambda_{k(j)}/|\lambda_{k(j)}| \to e^{-i\varphi}$. Meanwhile, the series (1) diverges at every point of exterior of $D(\Lambda,a)$ (except for the origin). Moreover, under the same conditions, by the Abel theorem (see [6], Theorem 3.1) for the series of the kind, the expansion (1) converges absolutely and uniformly on every compact subset of $D(\Lambda,a)$.

The problem of describing the set of singular points for $g_{\Lambda,a}$ on the boundary $\partial D(\Lambda,a)$ counts a long history. It originates in the investigation (started as early as the 19th century) of domains of existence for functions representable by power series. In this regard, we note the works of J. Hadamard [4] and E. Fabry [2]. In the works of G. Pólya [17], W. Fuchs [3] and P. Malliavin [16] the following result was obtained. The necessary and sufficient condition of existence for each sum of the Taylor series, converging in the unit disk, of a singular point on any arc of the boundary of this disk of length $2\pi\tau$ is $\tau = n^0(\Lambda)$ (Λ is a sequence indexes of Taylor series with nonzero coefficients). G. Pólya (see [18], [19]), Carlson and Landau (see [18], Chapter II, § 5.2) and V. Bernstein [1] extended this result to the case of Dirichlet series. In work [7] the result is obtained for these series whose special cases are all specified results for Taylor and Dirichlet series. It was proved that each sum of the Dirichlet series has a singular point on the segment of length $2\pi\tau$ lying on the convergence line, then and only then, when $\tau = n^0(\Lambda)$ and $S_\Lambda = 0$ (we will define the index of condensation S_Λ below). The singular points for the general series (1) and the series of exponents are studied in [8], [15].

This paper studies singular points for the sum of series (1) on arcs of the boundary $\partial D(\Lambda,a)$ of the following form. Let γ be the arc of the boundary connecting points z_1 and z_2. The arc γ will be called an R-arc if $|z_2 - z_1| = R$.

Let Λ be a regular sequence, i.e., it is a part of a regularly distributed set (see [15], Chapter II). It follows from Theorem 4.1 [8] that in this case, under certain restrictions on $D(\Lambda,a)$, each $g_{\Lambda,a}$ has a singular point on any R-arc if and only if $S_\Lambda = 0$. Λ is regular if and only if the maximal density $n^0(\Lambda,\psi,\varphi)$ (in the angle bounded by the rays $re^{i\psi}$, $re^{i\varphi}$) does not exceed the length of the corresponding arc of the boundary of a convex compact set (see [13], Theorem 1). Thus, sufficient conditions for existence of singular points for $g_{\Lambda,a}$ on R-arcs are the special boundedness of maximal density $n^0(\Lambda,\psi,\varphi)$ and the equality $S_\Lambda = 0$. In this research it is shown that in the general case these conditions are also necessary for the existence of singular points for $g_{\Lambda,a}$ on R-arcs. Moreover, for one class of the sequences Λ (which are concentrated along some ray $re^{i\varphi}$) it is proved that these conditions give a criterion for the existence of singular points for $g_{\Lambda,a}$ on R-arcs. All the results on Taylor and Dirichlet series, which are marked above, and the previously mentioned result from [7] are particular cases of this statement.

First of all, we study the influence of some characteristics of Λ on the presence of singular points for $g_{\Lambda,a}$ on R-arcs. We assume (see [9], [11])

$$S^0_\Lambda = \liminf_{\delta\to 0} \frac{S_\Lambda(\delta)}{\delta}. \quad S_\Lambda = \lim_{\delta\to 0} S_\Lambda(\delta), \quad S_\Lambda(\delta) = \liminf_{k\to\infty} \frac{\ln |q^k_\Lambda(\lambda_k,\delta)|}{|\lambda_k|},$$

$$q^k_\Lambda(z,\delta) = \prod_{\lambda_m \in B(\lambda_k,\delta|\lambda_k|), m\neq k} \left(\frac{z-\lambda_m}{3\delta|\lambda_m|}\right)^{n_m}.$$

Let $n_\Lambda(z,\delta)$ be the number of points $\lambda_k \in B(z,\delta|z|)$ with their multiplicities n_k taken into account.

Theorem 1. *Let $\Lambda = \{\lambda_k,n_k\}$, $m(\Lambda) = 0$ and $S_\Lambda^0 = -\infty$. Then for each $R > 0$ there exists a sequence $a \in \mathcal{U}(\Lambda)$ such that function $g_{\Lambda,a}$ has no singular points on some R-arc of boundary $\partial D(\Lambda,a)$.*

Proof. $R > 0$ and number $\delta_0 \in (0,1/15)$ satisfies the condition $\psi \in (0,\pi/4)$ if $\psi \in (0,\pi/2)$ and $|e^{i\psi}-1| \leqslant 2\delta_0(1-\delta_0)^{-1}$. Since $S_\Lambda^0 = -\infty$, according to the definition of S_Λ^0 we find $\delta \in (0,\delta_0)$ and the sequence $\{\lambda_{k(p)}\}$ such that

$$\ln|q_\Lambda^{k(p)}(\lambda_{k(p)},\delta)| \leqslant -\beta|\lambda_{k(p)}|, \quad p \geqslant 1, \quad \beta = 12R\delta. \tag{4}$$

Passing to the subsequence, we can assume that

$$\lambda_{k(p)}/|\lambda_{k(p)}| \to e^{-i\varphi}, \quad p \to \infty, \quad |\lambda_{k(p+1)}| \geqslant 2|\lambda_{k(p)}|, \quad p \geqslant 1. \tag{5}$$

The function $g_{\Lambda,a}$ is found as the sum of the series

$$g(z) = \sum_{p=1}^{\infty} c_p g_p(z). \tag{6}$$

To do this, we construct an auxiliary sequence $\Lambda_2 \subset \Lambda$, $\Lambda_2 = \cup_{p\geqslant 1}\Lambda_{2,p}$. Let $B_p(\alpha) = B(\lambda_{k(p)}, \alpha\delta|\lambda_{k(p)}|)$. By (5) and inequality $\delta_0 < 1/15$, disks $B_p(1)$, $p \geqslant 1$, do not intersect in pairs. We fix $p \geqslant 1$. The set $\Lambda_{2,p}$ is formed from multiple points $\lambda_k \in B_p(1)$. If

$$n_\Lambda(\lambda_{k(p)},\delta) - n_{k(p)} < \beta|\lambda_{k(p)}| + 1, \tag{7}$$

then, $\Lambda_{2,p}$ pair $\lambda_{k(p)}$,1 is taken as and all pairs λ_k, n_k such that $k \neq k(p)$ and $\lambda_k \in B_p$. In this case we assume $m_p = n_\Lambda(\lambda_{k(p)},\delta) - n_{k(p)} + 1$. Let now

$$n_\Lambda(\lambda_{k(p)},\delta) - n_{k(p)} \geqslant \beta|\lambda_{k(p)}| + 1. \tag{8}$$

Then we reduce the multiplicity $\lambda_{k(p)}$ to 1 and from the disk $B_p(1)$ we withdraw as many points λ_k without taking $\lambda_{k(p)}$, or we reduce their multiplicities n_k (without changing their designations) that inequalities

$$\beta|\lambda_{k(p)}| \leqslant m_p - 1 < \beta|\lambda_{k(p)}| + 1, \tag{9}$$

are satisfied, where m_p is the number of remaining points λ_k with taking into account their multiplicities. In this case, as $\Lambda_{2,p}$ all the remaining pairs λ_k,n_k are taken. Thus the sequence $\Lambda_2 \subset \Lambda$ is constructed. We will show that $\bar{n}(\Lambda_2) < \infty$. Let $r > 0$ and $p(r)$ be the maximal number of disk $B_p(1)$ having a non-empty intersection with $B(0,r)$. Then $r \geqslant \geqslant |\lambda_{k(p)}|(1-\delta)$ and by (7), (9) we obtain:

$$\frac{n(r,\Lambda)}{r} \leqslant \frac{n(|\lambda_{k(p(r))}|(1+\delta),\Lambda)}{|\lambda_{k(p(r))}|(1-\delta)} \leqslant \sum_{p=1}^{p(r)} \frac{m_p}{|\lambda_{k(p(r))}|(1-\delta)} \leqslant$$
$$\leqslant 2\sum_{p=1}^{p(r)} \frac{\beta|\lambda_{k(p)}|+1+1}{|\lambda_{k(p(r))}|} \leqslant 2\sum_{p=1}^{p(r)} \frac{|\lambda_{k(p)}|}{|\lambda_{k(p(r))}|}\left(\beta+\frac{2}{|\lambda_{k(p)}|}\right).$$

Using inequality (5), we get:

$$\frac{n(r,\Lambda)}{r} \leqslant C\sum_{p=1}^{p(r)} \frac{|\lambda_{k(p)}|}{|\lambda_{k(p(r))}|} \leqslant C\sum_{p=1}^{p(r)} \frac{1}{2^{p(r)-p}} \leqslant 2C.$$

It follows that $\bar{n}(\Lambda_2) < \infty$. We show now that inequalities (4) are not violated if Λ is replaced by Λ_2. If inequality (7) is true then by construction

$$q_{\Lambda}^{k(p)}(\lambda_{k(p)},\delta) = q_{\Lambda_2}^{k(p)}(\lambda_{k(p)},\delta).$$

Suppose now that inequality (8) is true. Then inequality (9) is true, and according to the definition of $q_{\Lambda_2}^k$, we have:

$$\ln|q_{\Lambda_2}^{k(p)}(\lambda_{k(p)},\delta)| = \sum_{\lambda_m\in B_p(1),m\neq k} n_m \ln\frac{|\lambda_{k(p)}-\lambda_m|}{3\delta|\lambda_m|} \leqslant$$
$$\leqslant \sum_{\lambda_m\in B_p(1),m\neq k} n_m \ln\frac{|\lambda_{k(p)}|}{3|\lambda_m|} \leqslant$$
$$\leqslant -\ln(3(1-\delta))(m_p-1) \leqslant m_p - 1 \leqslant -\beta|\lambda_{k(p)}|, \quad p \geqslant 1.$$

Thus,

$$\ln|q_{\Lambda_2}^{k(p)}(\lambda_{k(p)},\delta)| \leqslant -\beta|\lambda_{k(p)}|. \tag{10}$$

Let us now define the function g_p, $p \geqslant 1$, by the formula

$$g_p(z) = \frac{1}{2\pi i}\int_{\partial B_p(5)} \frac{e^{\lambda z}d\lambda}{\tau_p(\lambda-\lambda_{k(p)})q_{\Lambda_2}^{k(p)}(\lambda,\delta)},$$

where we define numbers $\tau_p \geqslant 1$ below. We get estimates from above on g_p. We have:

$$\frac{|\lambda - \lambda_k|}{3\delta|\lambda_k|} \geqslant 1, \quad \lambda \notin B_p(5), \quad \lambda_k \in B_p(1).$$

Since $\tau_p \geqslant 1$, we obtain:

$$|g_p(z)| \leqslant \sup_{\lambda \in \partial B_p(5)} |e^{\lambda z}| \leqslant \exp(\mathrm{Re}(\lambda_{k(p)} z) + 5\delta|\lambda_{k(p)}||z|), \quad z \in \mathbb{C}. \quad (11)$$

Let us now define the coefficients of c_p. Let

$$c_p = \exp(-\beta|\lambda_{k(p)}|), \quad p \geqslant 1. \quad (12)$$

We will show that series (6) converges on compact sets K in the domain $D = B(0, 2R) \cap \{z : \mathrm{Re}(ze^{-i\varphi}) < 2R\delta\}$ uniformly. We take $\varepsilon > 0$ such that

$$\mathrm{Re}(ze^{-i\varphi}) < R\delta - 2\varepsilon, \quad z \in K.$$

By (5) there is a number p_0 such that

$$\mathrm{Re}(\lambda_{k(p)} z) < (\mathrm{Re}(ze^{-i\varphi}) + \varepsilon)|\lambda_{k(p)}|, \quad z \in K, \quad p \geqslant p_0.$$

Then, we get for $z \in K$ by (11) and (12):

$$\sum_{p=p_0}^{\infty} |c_p g_p(z)| \leqslant \sum_{p=p_0}^{\infty} c_p \exp\left((R\delta - \varepsilon + 5\delta R)|\lambda_{k(p)}|\right) \leqslant \sum_{p=p_0}^{\infty} e^{-\varepsilon|\lambda_{k(p)}|}.$$

Since $\bar{n}(\Lambda_2) < \infty$ then the last series converges. Thus, the function g is analytical in the domain D for any $\tau_p \geqslant 1$.

Since $\delta \in (0, 1/15)$ and the points $\lambda_k \in \Lambda_2$ belong to the disks $B_p(1)$, then by (5) there is $\psi \in (0, \pi/2)$ such that $|e^{i\psi} - 1| \leqslant 2\delta(1-\delta)^{-1}$ and $\Theta(\Lambda_2) \subset \{e^{i\theta}, \theta \in (\varphi - \psi, \varphi + \psi)\}$ holds true. We will show that for any $\tau_p \geqslant 1$ the function g is represented by series (5) which converges uniformly on compact sets in the angle

$$\Gamma = \left\{z : \mathrm{Re}(ze^{-i(\varphi+\psi)}) < 0\right\} \cap \left\{z : \mathrm{Re}(ze^{-i(\varphi-\psi)}) < 0\right\}.$$

Using the residue calculus and the definition of the function $q_{\Lambda_2}^{k(p)}$, for every $p \geqslant 1$ we have

$$g_p(z) = \frac{b_{k(p),0}}{\tau_p} e^{\lambda_{k(p)} z} + \sum_{\lambda_k, n_k \in \Lambda_{2,p}, k \neq k(p)} \sum_{n=0}^{n_k - 1} \frac{b_{k,n}}{\tau_p} z^n e^{\lambda_k z},$$

where $b_{k(p),0} = \left(q^{k(p)}_{\Lambda_2}(\lambda_{k(p)}, \delta)\right)^{-1}$. Let $b_{k(p),n} = 0$, $n = \overline{1,n_{k(p)} - 1}$. We define the coefficients $\{a_{k,n}\}$:

$$a_{k,n} = 0, \quad \lambda_k, n_k \notin \Lambda_2, \quad a_{k,n} = \frac{c_p b_{k,n}}{\tau_p}, \quad \lambda_k, n_k \in \Lambda_{2,p}, \quad p \geqslant 1.$$

Since the disks $B_p(1)$, $p \geqslant 1$ do not intersect in pairs, the definition is well-defined. Let us find the numbers τ_p, $p \geqslant 1$. By (10) and (12) we have:

$$\max_{\lambda_k, n_k \in \Lambda_{2,p}} \ln |c_p b_{k,n}| \geqslant \ln |c_p b_{k(p),0}| \geqslant \ln c_p + \beta|\lambda_{k(p)}| = 0.$$

We choose the numbers $\tau_p \geqslant 1$ such that

$$\max_{\lambda_k, n_k \in \Lambda_{2,p}} \ln |a_{k,n}| = \max_{\lambda_k, n_k \in \Lambda_{2,p}} \ln |c_p b_{k,n}| - \ln a_p = 0. \qquad (13)$$

We find the convergence domain of series (1) with the coefficients $a_{k,n}$ defined above. Since $\bar{n}(\Lambda_2) < \infty$, $m(\Lambda) = m(\Lambda_2) = 0$ then the Cauchy–Hadamard and Abel theorems (see [6]) show that series (1) converges uniformly on compact sets in the convex domain $D(\Lambda,a)$, defined by formulas (2) and (3), and diverges at each point of its exterior (except for the origin of coordinates). By (13) and (3)

$$h(\theta, a, \Lambda) \geqslant \liminf_{k\to\infty} \min_{0\leqslant n\leqslant n_k-1} \frac{\ln(1/|a_{k,n}|)}{|\lambda_k|} = \\ = \limsup_{p\to\infty} \max_{\lambda_k, n_k \in \Lambda_{2,p}} \frac{\ln |a_{k,n}|}{|\lambda_k|} = 0$$

for all $e^{i\theta} \in \Theta(\Lambda_2)$. In addition, there are numbers $s(p)$, $n(p)$ such that the pair $\lambda_{s(p)}$, $n_{s(p)}$ is an element of the sequence Λ_2 and

$$\ln |a_{s(p),n(p)}| = 0, \quad p \geqslant 1. \qquad (14)$$

We assume that $\lambda_{s(p)}/|\lambda_{s(p)}| \to e^{-i\rho}$, $p \to \infty$. Then $e^{i\rho} \in \Theta(\Lambda_2)$. And we can assume $\rho = \varphi + \alpha$, where $\alpha \in (-\psi, \psi)$. By (14) we have:

$$h(\varphi + \alpha, a, \Lambda) \leqslant \liminf_{p\to\infty} \frac{\ln(1/|a_{s(p),n(p)}|)}{|\lambda_{s(p)}|} = 0.$$

Thus, series (1) diverges at each point of $\Pi = \{z : \mathrm{Re}(ze^{-i(\varphi+\alpha)}) > 0\}$. In addition, it converges uniformly on compact sets in the angle $\Gamma \subset D_0(a,\Lambda)$, along with series (6). Therefore, the function $g_{\Lambda,a} = g$ is analytical in the domain $G = D \cup \Gamma$.

Let $\alpha \geqslant 0$ and $w \in \partial B(0, 2R) \cap (L_0 = \{z : \mathrm{Re}(ze^{-i\varphi}) = 0\})$, $\mathrm{Im}(we^{-i\varphi}) > 0$ (the case $\alpha < 0$ is similar). Straight lines that are perpendicular $L_3 = \{z : \mathrm{Re}(ze^{-i(\varphi+\psi)}) = 0\}$ and pass through the points 0 and w, are denoted by L_1 and L_2. Since $\psi \in (0, \pi/4)$, the distance between these lines is strictly greater than R. Let Ω be the area bounded by the lines L_0, L_1, L_2. It lies in the domain G. Also, some neighborhood V of the interval $(0,w) \subset L_0 \cap \partial\Omega$ lies in the domain G. Thus, the function $g_{\Lambda,a}$ is analytical in the domain $\Omega \cup V$.

Since $\psi \in (0, \pi/4)$ then the half-string which is limited by lines L_1, L_2, L_3 lies in the angle $\Gamma \subset D(\Lambda,a)$. Series (1) diverges in Π. Therefore, there exists an R-arc which lies in the intersection of the domain $D \cup V$ and the half-plane $\{z : \mathrm{Re}(ze^{-i(\varphi+\alpha)}) \leqslant 0\}$. By construction, $g_{\Lambda,a}$ has no singular points on this arc. $\square$

Example 1. Let $\Lambda = \{\lambda_k, n_k\}$, $n_k = 1$ and $\lambda_{2k} = k$, $\lambda_{2k-1} = k - e^{-\varepsilon k}$, $k \geqslant 1$, where $\varepsilon > 0$. Then $m(\Lambda) = 0$. If $\delta \in (0, 1/3)$, then

$$|q_\Lambda^{2k}(\lambda_{2k},\delta)| \leqslant \left| \left(\frac{\lambda_{2k} - \lambda_{2k-1}}{3\delta|\lambda_{2k-1}|} \right) \right| \leqslant \frac{e^{-\varepsilon k}}{3\delta(k - e^{-\varepsilon k})}.$$

Therefore,

$$S_\Lambda = \lim_{\delta\to 0} \liminf_{k\to\infty} \frac{\ln |q_\Lambda^l(\lambda_l,\delta)|}{|\lambda_l|} \leqslant \lim_{\delta\to 0} \liminf_{k\to\infty} k^{-1} \ln \left(\frac{e^{-\varepsilon k}}{(3\delta(k - e^{-\varepsilon k})} \right) = -\varepsilon.$$

By the last inequality it follows that $S_\Lambda^0 = -\infty$. We also note that the work [20], Chapter II, § 4, has a nontrivial example of a sequence Λ for which $S_\Lambda = 0$ and $S_\Lambda^0 = -\infty$.

Theorem 2. *Let $\Lambda = \{\lambda_k, n_k\}$ and $m(\Lambda) \neq 0$. Then for every $R > 0$ there exists a sequence $a \in \mathcal{U}(\Lambda)$ such that the function $g_{\Lambda,a}$ has no singular points on any R-arc of the boundary $\partial D(\Lambda,a)$.*

Proof. Let $R > 0$, $\sigma \in (0,1]$ and

$$D_\sigma = \{z : \mathrm{Re} z < \sigma\} \cap \{z : |\mathrm{Im} z| < \sigma R\}, \quad T_\sigma = \{z : \mathrm{Re} z = \sigma, |\mathrm{Im} z| \leqslant \sigma R\}.$$

The segment T_σ lies on the boundary of the half-string D_σ. We choose $\sigma \in (0,1)$ such that $(1-\sigma)^2 + \sigma R^2 < 1$ and $\sigma < (2\sqrt{3}R)^{-1}$. Then T_σ lies in disk $B(1,1)$, the domain $G_1 = D_\sigma \setminus \{z : \mathrm{Re} z < -2^{-1}\}$ lies in the disk $B(0,1)$, and $D_\sigma \setminus G_1$ lies in the truncated angle

$$\Gamma = \{\mathrm{Re}(ze^{i\pi/6}) < 0\} \cap \{\mathrm{Re}(ze^{-i\pi/6}) < 0\} \cap \{z : \mathrm{Re} z < -2^{-1}\}.$$

Since T_σ is a compact set, then there exists an $\varepsilon \in (0,1)$ such that the rectangle $T_\sigma(\varepsilon) = D_\sigma \cap \{z : \mathrm{Re} z > \sigma - \varepsilon\}$ lies in the disk $B(1, r_0)$ for any $r_0 \in (e^{-1}, 1)$.

According to the condition $m(\Lambda) \neq 0$ there exists a sequence $\{\lambda_{k(p)}\}$ such that $\lambda_{k(p)}/|\lambda_{k(p)}| \to e^{-i\varphi}$ and $\sigma n_{k(p)}/|\lambda_{k(p)}| \geqslant \tau > 0$. We suppose $\mu_p = e^{i\varphi}\sigma^{-1}\lambda_{k(p)}$. Then $\mu_p/|\mu_p| \to 1$ and $n_{k(p)}/|\mu_p| \geqslant \tau$, $p \geqslant 1$. We assume that $|\mu_{p+1}| \geqslant 2|\mu_p|$, $p \geqslant 1$. Let $0 < \gamma \leqslant \min\{4^{-1}\varepsilon, 2^{-1}\tau, 8^{-1}\}$. We can also assume that

$$\gamma|\mu_p| \leqslant m(p) \leqslant \min\{4^{-1}\varepsilon|\mu_p|, n_{k(p)}\}, \quad p \geqslant 1, \tag{15}$$

where $m(p)$ are some positive natural numbers. By construction

$$\mathrm{Re}(\mu_p z) \leqslant |\mu_p|(\mathrm{Re} z - 2^{-1}\gamma \ln r_0), \quad z \in B(0,t), \quad p \geqslant p_t. \tag{16}$$

We suppose $c_p = \exp(-(\sigma + 4^{-1}\gamma \ln r_0)|\mu_p|)$, $p \geqslant 1$ and consider the series

$$g_\sigma(z) = \sum_{p=1}^{\infty} c_p (z-1)^{m(p)} e^{\mu_p z}, \tag{17}$$

We show that it converges uniformly on compact sets in the domain D_σ. Considering the embedding $T(\varepsilon) \subset B(1,r_0)$, by (15) and (16) we have:

$$|c_p(z-1)^{m(p)} e^{\mu_p z}| \leqslant \exp(\gamma|\mu_p| \ln r_0 - (\sigma - \mathrm{Re} z + 4^{-1}3\gamma \ln r_0)|\mu_p|)) \leqslant$$

$$\leqslant \exp(4^{-1}\gamma \ln r_0 |\mu_p|), \quad z \in T(\varepsilon), \quad p \geqslant p_1. \tag{18}$$

Since $G_1 \subset B(0,1)$, $r_0 > e^{-1}$ then by (15), (16) and the definition of $T(\varepsilon)$ we have:

$$|c_p(z-1)^{m(p)} e^{\mu_p z}| \leqslant \exp(m(p) \ln 2 -$$

$$(\sigma - \mathrm{Re} z + 4^{-1}3\gamma \ln r_0)|\mu_p|)) \leqslant \exp((-\varepsilon/2 + 3\gamma/4)|\mu_p|) \leqslant$$

$$\leqslant \exp(-5\varepsilon/16|\mu_p|), \quad z \in G_1 \setminus T(\varepsilon), \quad p \geqslant p_1. \tag{19}$$

Considering the embedding $D_\sigma \setminus G_1 \subset \Gamma$, we get:

$$|c_p(z-1)^{m(p)}e^{\mu_p z}| \leqslant \exp(m(p)\ln(1+|z|)-(\sigma-\mathrm{Re}z+4^{-1}3\gamma\ln r_0)|\mu_p|)) \leqslant$$

$$\leqslant \exp(|\mu_p||z|/4+(\mathrm{Re}z+3/32)|\mu_p|)) \leqslant \exp((|\mathrm{Re}z|/2+\mathrm{Re}z+3/32)|\mu_p|) \leqslant$$

$$\leqslant \exp(-5/32|\mu_p|), \quad z \in (D_\sigma \setminus G_1)\cap B(0,t), \quad p \geqslant p_t.$$

By the last inequality and (18), (19) it follows that series (17) converges uniformly on the compact sets of the domain D_σ. Therefore, $g_\sigma \in H(D_\sigma)$. Opening brackets in (17), we obtain:

$$g_\sigma(z) = \sum_{p=1,n=0}^{\infty,m(p)} b_{p,n}z^n e^{\mu_p z} = \sum_{k=1,n=0}^{\infty,n_k-1} a_{k,n}w^n e^{\lambda_k w}, \tag{20}$$

where $z = \sigma e^{-i\varphi}w$, $a_{k,n} = 0$, if $k \neq k(p)$ or $k = k(p)$ and $n > m(p)$. Since $r_0 < 1$ then $G_0 = \{z : \mathrm{Re}z > \sigma + (\gamma\ln r_0)/8\} \cap D_\sigma \neq \emptyset$. Similarly to (16), we obtain:

$$\mathrm{Re}(\mu_p z) \geqslant |\mu_p|(\mathrm{Re}z + (\gamma\ln r_0)/8), \quad z \in G_0, \quad p \geqslant p_0.$$

It is easy to notice that $|b_{p,0}| = c_p$. Therefore, taking into account the definition of c_p we have:

$$|b_{p,0}||e^{\mu_p z}| \geqslant 1, \quad z \in G_0, \quad p \geqslant p_0.$$

Thus, the first series in (20) diverges at every point $z \in G_0$. Let

$$\Gamma_0 = \{\mathrm{Re}(ze^{i\pi/6}) < 0\} \cap \{\mathrm{Re}(ze^{-i\pi/6}) < 0\} \cap \{z : \mathrm{Re}z < -3\},$$

and $z \in \Gamma_0$. Coefficients $b_{p,n}$ are estimated above by $2^{m(p)}$. Therefore, taking into account (15), (16) and definitions c_p, r_0, ε, γ we have:

$$|b_{p,n}z^n e^{\mu_p z}| \leqslant \exp(m(p)+m(p)\ln(1+|z|)+(\mathrm{Re}z-4^{-1}3\gamma\ln r_0)|\mu_p|) \leqslant$$

$$\leqslant \exp((\varepsilon/4+\varepsilon/4|z|+3\gamma/4+\mathrm{Re}z)|\mu_p|) \leqslant \exp((1/2+\mathrm{Re}z/2)|\mu_p|) \leqslant e^{-|\mu_p|}.$$

It means that the first series in (20) converges uniformly on Γ_0.

It follows from the above that on a certain σR-arc, lying in the string $\{z : |\mathrm{Im}z| < \sigma R\}$ and on the boundary of the convergence domain of this series, its sum g_σ has no singular points. Then the sequence of coefficients of the second series in (20) is the required one. □

Example 2. Let $\Lambda = \{\lambda_k, n_k\}$, $\lambda_k = 2^k$ and $n_k = 2^{k-1}$, $k \geqslant 1$. Then $\bar{n}(\Lambda) \leqslant 1$ and $m(\Lambda) = 1/2$.

By theorems 1,2 conditions $S_\Lambda^0 > -\infty$, $m(\Lambda) = 0$ are necessary for the existence of singular points for all functions $g_{\Lambda,a}$, $a \in \mathcal{U}(\Lambda)$, on every R-arc of boundaries of the convergence domains corresponding to series (1). We show that these conditions are sufficient for one of the classes of sequences as well.

Lemma 1. *Let $\Lambda = \{\lambda_k, n_k\}$ such that $m(\Lambda) = 0$, $S_\Lambda^0 > -\infty$, and $\lambda_k/|\lambda_k| \to e^{-i\varphi}$, $k \to \infty$. Then $n^0(\Lambda) < \infty$.*

Proof. Let $\delta \in (0,1)$. As in Theorem 1, we obtain:

$$\ln|q_\Lambda^k(\lambda_k,\delta)| \leqslant -\ln(3(1-\delta))(n_\Lambda(\lambda_k,\delta) - 1).$$

Therefore, taking into account equality $m(\Lambda) = 0$ we obtain:

$$S_\Lambda^0 = \liminf_{\delta\to 0} \liminf_{k\to\infty} \frac{\ln|q_\Lambda^k(\lambda_k,\delta)|}{\delta|\lambda_k|} \leqslant \leqslant \liminf_{\delta\to 0} \liminf_{k\to\infty} \frac{-\ln(3(1-\delta))n_\Lambda(\lambda_k,\delta)}{\delta|\lambda_k|} = = -\limsup_{\delta\to 0} \ln(3(1-\delta)) \limsup_{k\to\infty} \frac{n_\Lambda(\lambda_k,\delta)}{\delta|\lambda_k|} = = -\limsup_{\delta\to 0} \limsup_{k\to\infty} \frac{n_\Lambda(\lambda_k,\delta)}{\delta|\lambda_k|}. \quad (21)$$

Let $n^0(\Lambda) > 0$. We choose a sequence $r_{p,\delta}$, $p \geqslant 1$ such that

$$\limsup_{r\to\infty} \frac{n(r,\Lambda) - n((1-\delta)r,\Lambda)}{\delta r} = = \lim_{p\to\infty} \frac{n(r_{p,\delta},\Lambda) - n((1-\delta)r_{p,\delta},\Lambda)}{\delta r_{p,\delta}} > 0, \quad (22)$$

$\delta \in (0,\delta_0)$. We assume that any ring $B(0, r_{p,\delta}) \setminus B(0, (1-\delta)r_{p,\delta})$ contains at least one λ_k. Let $U_{p,\delta}$ be the group of all points λ_k, belonging to it and $\lambda_{k(p,\delta)} \in U_{p,\delta}$. According to the condition $U_{p,\delta} \subset B(\lambda_{k(p,\delta)}, 4\delta|\lambda_{k(p,\delta)}|)$, $p \geqslant p(\delta)$. Therefore, taking into account (22) and (21) we obtain:

$$n^0(\Lambda) \leqslant \limsup_{\delta\to 0} \limsup_{p\to\infty} \frac{n_\Lambda(\lambda_{k(p,\delta)}, 4\delta)}{\delta r_p(\delta)} \leqslant \limsup_{\delta\to 0} \limsup_{p\to\infty} \frac{n_\Lambda(\lambda_{k(p,\delta)}, 4\delta)}{\delta|\lambda_{k(p,\delta)}|} =$$

$$= 4 \limsup_{\delta\to 0} \limsup_{p\to\infty} \frac{n_\Lambda(\lambda_{k(p,\delta)},\delta)}{\delta|\lambda_{k(p,\delta)}|} \leqslant 4 \limsup_{\delta\to 0} \limsup_{k\to\infty} \frac{n_\Lambda(\lambda_k,\delta)}{\delta|\lambda_k|} \leqslant -4S^0_\Lambda.$$

□

Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain, $W(\Lambda,D)$ be a closure in $H(D)$ (in the topology of uniform convergence on compact sets) of linear span $\mathcal{E}(\Lambda) = \{z^n e^{\lambda_k z}\}$. We need the criterion of fundamental principle from [11], Theorem 3.2. Let us express it in the particular case. We assume

$$L(\varphi,D) = \partial D \cap \{w : \mathrm{Re}(we^{-i\varphi}) = H_D(\varphi)\}, \quad H_D(\psi) = \sup_{z\in D} \mathrm{Re}(ze^{-i\psi})$$

is a supported function of the domain D, $\tau(\varphi, D)$ is the length of $L(\varphi, D)$ (possibly equal to zero).

Lemma 2. *Let $\Lambda = \{\lambda_k, n_k\}$ such that $\lambda_k/|\lambda_k| \to e^{-i\varphi}$, and D be a bounded convex domain. If $W(\Lambda, D)$ is non-trivial (i. e., the system $\mathcal{E}(\Lambda)$ is not complete in $H(D)$), then the following statements are equivalent:*

1) *$\mathcal{E}(\Lambda)$ is a basis in $W(\Lambda, D)$;*
2) *$S_\Lambda = 0$ and $n^0(\Lambda) \leqslant \tau(\varphi, D)/2\pi$.*

We note that the system $\mathcal{E}(\Lambda)$ is not complete in $H(D)$ when and only when there exists an entire function f of the exponential type, the shift of the conjugate diagram (see [15], Chapter I, § 5) of which lies in D.

Theorem 3. *Let $\Lambda = \{\lambda_k, n_k\}$ such that $\lambda_k/|\lambda_k| \to e^{-i\varphi}$, and $R \geqslant 0$; the following statements are equivalent:*

1) *Each function $g_{\Lambda,a}$ on every R-arc $\gamma \subset \partial D(\Lambda, a)$ has a singular point.*
2) *$S_\Lambda = 0$ and $n^0(\Lambda) \leqslant R/2\pi$.*

Proof. Let us assume that statement 2) holds true and $a \in \mathcal{U}(\Lambda)$. It is easy to notice that from inequality $n^0(\Lambda) < \infty$ relations $m(\Lambda) = 0$ and $\bar{n}(\Lambda) < \infty$ follow. Then, as in Theorem 1, domain $D(\Lambda, a)$ is defined by formula (2). According to condition $\Theta(\Lambda) = \{e^{i\varphi}\}$. Hence, $D(\Lambda, a) = \{z : \mathrm{Re}(ze^{-i\varphi}) < c\}$, and any R-arc of the boundary $\partial D(\Lambda, a)$ is a segment $[z_1, z_2] \subset \{z : \mathrm{Re}(ze^{-i\varphi}) = c\}$ of length R. Let's assume that $g_{\Lambda,a}$ has no singular points on $[z_1, z_2]$.

Then there exists $\delta > 0$ such that the function $g_{\Lambda,a}$ is analytical in the domain $\Omega = K + B(0,3\delta)$, where K is a square with the side $[z_1, z_2]$, lying in the closure $D(\Lambda, a)$. By Lemma 2.1 from work [12] Λ is a part of the sequence $\Lambda_1 = \{\mu_p, m_p\}$ having density $n(\Lambda_1) = n^0(\Lambda)$. It can be assumed that $\mu_p/|\mu_p| \to e^{-i\varphi}$ (arguments μ_p do not affect the density Λ_1). We get

$$f(z) = \prod_{p\geqslant 1} \left(1 - \frac{z^2}{(\mu_p)^2}\right)^{m_p}.$$

The function f is entire and has the exponential type, its conjugate diagram coincides with the segment $[-\alpha, \alpha]$, where $\alpha = \pi n^0(\Lambda)e^{i(\varphi+\pi/2)}$ (see [13], Chapter 2, §1, Theorem 2). The shift of this segment lies in the domain $G = K + B(0,\delta) - \delta e^{i\varphi} \subset D(\Lambda, a)$. Hence, $\mathcal{E}(\Lambda)$ is not complete in $H(G)$.

Since $g_{\Lambda,a}$ is represented by the series (1) in the domain $D(\Lambda, a)$ then $g_{\Lambda,a} \in W(\Lambda, G)$. In addition, $g_{\Lambda,a}$ is analytical in the domain $G + B(0,\delta) \subset \Omega$. Thus, the conditions of Theorem 12.1 from [5] on the continuation of the spectral synthesis are fulfilled. Therefore, according to it, $g_{\Lambda,a} \in W(\Lambda, G + B(0,\delta))$.

As we have shown above, $\mathcal{E}(\Lambda)$ is not complete in $H(G+B(0, \delta))$. According to statement 2) and taking into account the construction we have: $S_\Lambda = 0$ and $n^0(\Lambda) \leqslant R/2\pi \leqslant \tau(\varphi, G + B(0, \delta))/2\pi$. Then, by Lemma 2, the function $g_{\Lambda,a}$ is represented by the series (1) in the domain $G+B(0,\delta)$. Thus, we have two representations of the function $g_{\Lambda,a}$ by the series (1) in the domain $G \subset D(\Lambda,a) \cap (G + B(0,\delta))$. Since $\mathcal{E}(\Lambda)$ is not complete in $H(G)$, then (see [15], Chapter 2, § 6, Theorem 6.2) these representations are the same. According to the definition of $D(\Lambda, a)$ it means that there exists an embedding $G + B(0, \delta) \subset D(\Lambda, a)$. By construction, however, this embedding is incorrect. We have a contradiction. Hence, $g_{\Lambda,a}$ has at least one singular point on the segment $[z_1, z_2]$.

Let us assume that statement 1) holds true. Then by Theorem 1 inequality $S^0_\Lambda > -\infty$ holds true. Referring to the definition of the quantities S^0_Λ and S_Λ we get $S_\Lambda = 0$. Theorem 2 implies that the equality $m(\Lambda) = 0$ is also true. Then, by Lemma 1 we get $n^0(\Lambda) < \infty$. It remains to prove that $n^0(\Lambda) \leqslant R/2\pi$. Let us assume the opposite: $\rho = n^0(\Lambda) > R/2\pi$. We choose $\varepsilon > 0$ such that $\rho - \varepsilon > R/2\pi$.

We suppose $L_1 = \{z \in \mathbb{C} : \operatorname{Re} z = 1\}$, $L_2 = \{z \in \mathbb{C} : \operatorname{Re} z = -1\}$. Let L_3 (L_4) be a straight line passing through the points with coordinates $(0, i\pi\rho)$ and $(1, i\pi(\rho - \varepsilon))$ ($(0, -i\pi\rho)$ and $(1, -i\pi(\rho - \varepsilon))$). We suppose $D = e^{i\varphi}D_0$, where D_0 is the domain bounded by straight lines L_1, L_2, L_3, L_4. It is an isosceles trapezoid. One of its bases of length $2\pi(\rho - \varepsilon)$

lies on the line L_1 and the other base of length $> 2\pi\rho$ lies on the line L_2.

The vertical segment with length $2\pi\rho$ lies in D_0 by construction. Therefore, the domain D contains a shift of a conjugate diagram of the function f. It means that $\mathcal{E}(\Lambda)$ is not complete in $H(D)$. We consider the subspace $W(\Lambda, D)$. We have $n^0(\Lambda) = \rho > \rho - \varepsilon = \tau(\varphi, D)/2\pi$. Then there exists a function $g \in W(\Lambda,D)$ by Lemma 2 which is not represented by the series (1) uniformly convergent on a compact set in the domain D.

Let us consider the domain $D_1 = e^{i\varphi}D_{0,1}$, where $D_{0,1} = D_0 \cap \{z \in \mathbb{C} : \mathrm{Re}z < 0\}$. It is an isosceles trapezoid, one of its bases coincides with the corresponding base of the trapezoid D, and the other base coincides with the segment $e^{i\varphi}\,[-i\pi\rho, i\pi\rho]$. The domain D_1 contains a shift of a conjugate diagram of the function f by construction. Therefore, $\mathcal{E}(\Lambda)$ is not complete in $H(D_1)$. We have $S_\Lambda = 0$ and $n^0(\Lambda) = \rho = \tau(\varphi,D_1)/2\pi$. Then the function g is represented by the series (1) in the domain D_1 by Lemma 2.

Since $n^0(\Lambda) < \infty$, this series converges in some half-plane $\{z : \mathrm{Re}(ze^{-i\varphi}) < c\}$ $(c \geqslant 0)$ and diverges at each point of its exterior, as in the beginning of the proof. The inequality $c < 1$ holds true. Indeed, otherwise the function g may be represented by the series (1) which converges in the domain D, which is impossible owing to choosing g.

By construction the function g is analytical in the domain D which crosses the line $\{z : \mathrm{Re}(ze^{-i\varphi}) = c\}$ at intervals of length strictly greater than $2\pi(\rho - \varepsilon)$. Thus, given the choice of the number $\varepsilon > 0$ we have the sum of the series (1), which has no singular points on a certain R-arc (a segment with length R) of the boundary of its convergence domain. We come to a contradiction with 1). Therefore, the assumption that $n^0(\Lambda) > R/2\pi$ is incorrect. □

Example 3. Let $\Lambda = \{\lambda_k, n_k\}$, $\lambda_k = kh$ and $n_k = 1$, $k \geqslant 1$. Then $S_\Lambda = 0$ (see [12], § 2) and $n^0(\Lambda) = h$.

The particular cases of Theorem 3 are all the above mentioned results for Dirichlet and Taylor series.

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