Probl. Anal. Issues Anal. Vol. 6 (24), No. 2, 2017, pp. 57–80
DOI: 10.15393/j3.art.2017.3850

UDC 517.98, 514.745.4

Ya. A. Kopylov

ORLICZ SPACES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS: DUALITY AND COHOMOLOGY

Abstract. We consider Orlicz spaces of differential forms on a Riemannian manifold. A Riesz-type theorem about the functionals on Orlicz spaces of forms is proved and other duality theorems are obtained therefrom. We also extend the results on the Hölder-Poincaré duality for reduced $L_{q,p}$-cohomology by Gol′dshtein and Troyanov to $L_{\Phi_I,\Phi_{II}}$-cohomology, where Φ_I and Φ_{II} are N-functions of class $\Delta_2 \cap \nabla_2$.

Key words: *Riemannian manifold, differential form, exterior differential, Orlicz space, Orlicz cohomology*

2010 Mathematical Subject Classification: *58A12, 46E30*

Introduction. This article is devoted to the study of the dual spaces of Orlicz spaces of differential forms on an oriented Riemannian manifold X.

L_p-theory of differential forms on Riemannian manifolds has been the subject of many papers and several books since the beginning of the 1980s. In 1976, Atiyah defined L_2-cohomology for a Riemannian manifold and initiated various applications of L_2-methods to the study of noncompact manifolds and quotient spaces of Riemannian manifolds by discrete groups of isometries. The L_2-cohomology of such manifolds was studied by Gromov, Cheeger–Gromov and others (see, for example, [2, 3, 12]). In the 1980's, Goldshtein, Kuz′minov, and Shvedov defined the L_p-de Rham complex on a Riemannian manifold M for arbitrary $p \in [1, \infty]$ and began to investigate its cohomology, which they called the L_p-cohomology of M; they obtained many results concerning the density of smooth forms in L_p (see, for example, [5]); the nontriviality and the Hausdorff property of L_p-cohomology on important classes of manifolds (see, for instance, [7, 8, 17]), duality for L_p-related spaces of differential forms and the induced duality for L_p-cohomology in [6]; compactly-supported approximation of L_p-forms (see, for example, [16]). In studying the asymptotic invariants of infinite groups and manifolds with pinched negative curvature, Gromov and Pansu also considered L_p-differential forms and l_p-simplicial cochains (see [12, 18, 19]). Gol′dstein and Troyanov obtained deep results about the L_{qp}-cohomology of Riemannian manifolds for $q \neq p$ in [9, 10, 11].

Like Orlicz function spaces, the Orlicz spaces L^Φ of differential forms are a natural nonlinear generalization of the spaces L^p. Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [13] and then by Agarwal, Ding, and Nolder in [1] (see also [4, 14]). In [13], Iwaniec and Martin established a Riesz-type theorem for an Orlicz space of differential forms on a domain in $\mathbb{R}^n$. Orlicz spaces of differential forms on a Riemannian manifold were apparently first examined by Panenko and the author in [15], where de Rham regularization operators were introduced and studied for Orlicz spaces of differential forms.

We prove a Riesz-type theorem for Orlicz spaces of differential forms on a Riemannian manifold and then, using it, describe the dual spaces of Orlicz–Sobolev-type spaces of differential forms, thus generalizing the results of Gol′dshtein, Kuz′minov, and Shvedov obtained in [6] for L^p-related spaces. The so-obtained results are applied for establishing the Hölder–Poincaré duality for the reduced Orlicz cohomology of X, which extends the Hölder–Poincaré duality for $L_{q,p}$-cohomology proved by Gol′dshtein and Troyanov in [11].

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we give the definition of Orlicz spaces of differential forms on a Riemannian manifold. The Riesz-type theorem for Orlicz spaces of differential forms (Theorem 3.1) is the contents of Section 3. Then, in Section 4, we examine the structure of the dual spaces to some L^Φ-related spaces of differential forms. Finally, in Section 5, we establish a theorem on the Poincaré duality for the $L_{\Phi_I,\Phi_{II}}$-cohomology of an oriented Riemannian manifold (Theorem 5.8).

1. N-functions and Orlicz function spaces.

Definition 1.1.

A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an N-function if

(i) Φ *is even and convex;*

(ii) $\Phi(x) = 0 \Longleftrightarrow x = 0$;
(iii) $\lim\limits_{x \to 0} \frac{\Phi(x)}{x} = 0$; $\quad \lim\limits_{x \to \infty} \frac{\Phi(x)}{x} = \infty$.

An N-function Φ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [20, Theorem 1, p. 7]). The left derivative φ of Φ is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim\limits_{t \to \infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 \,:\, \varphi(t) > s\}, \quad s > 0,$$

is called the *left inverse* of φ.

The functions Φ, Ψ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$

are called *complementary N-functions.*

The N-function Ψ complementary to an N-function Φ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) \,:\, x \geq 0\}, \quad y \in \mathbb{R}.$$

N-functions are classified in accordance with their growth rates as follows:

Definition 1.2. *An N-function Φ is said to satisfy the Δ_2-condition for large x (for small x, for all x), which is written as $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2(0)$, or $\Phi \in \Delta_2$), if there exist constants $x_0 > 0$, $K > 2$ such that $\Phi(2x) \leq$ $\leq K\Phi(x)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$); and it satisfies the ∇_2-condition for large x (for small x, or for all x), which is denoted symbolically as $\Phi \in \nabla_2(\infty)$ ($\Phi \in \nabla_2(0)$, or $\Phi \in \nabla_2$) if there are constants $x_0 > 0$ and $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$).*

Henceforth, let Φ be an N-function and let (Ω, Σ, μ) be a measure space.

Definition 1.3. *The set $\tilde{L}^\Phi = \tilde{L}^\Phi(\Omega) = \tilde{L}^\Phi(\Omega, \Sigma, \mu)$ is defined to be the set of measurable functions $f : \Omega \to \mathbb{R}$ such that*

$$\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu < \infty.$$

Definition 1.4. *The linear space*

$$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \\ = \{f : \Omega \to \mathbb{R} \text{ measurable } : \rho_\Phi(af) < \infty \text{ for some } a > 0\}$$

is called an Orlicz space on (Ω, Σ, μ).

The corresponding *Morse–Transue space* is the space

$$M^\Phi = M^\Phi(\Omega) = M_\Phi(\Omega, \Sigma, \mu) = \\ = \{f : \Omega \to \mathbb{R} \text{ measurable } : \rho_\Phi(af) < \infty \text{ for } \textit{all } a > 0\}.$$

For an Orlicz space $L^\Phi = L^\Phi(\Omega, \Sigma, \mu)$, the N-function Φ is called Δ_2-*regular* if $\Phi \in \Delta_2(\infty)$ when $\mu(\Omega) < \infty$ or $\Phi \in \Delta_2$ when $\mu(\Omega) = \infty$ or $\Phi \in \Delta_2(0)$ for μ the counting measure on countable Ω.

Let Ψ be the complementary N-function to Φ.

Below we as usual identify two functions equal outside a set of measure zero.

If $f \in L^\Phi$ then the functional $\|\cdot\|_\Phi$ (called *the Orlicz norm*) defined by

$$\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup\left\{\left|\int_\Omega fg\,d\mu\right| : \rho_\Psi(g) \le 1\right\}$$

is a seminorm. It becomes a norm if μ satisfies the *finite subset property* (see [20, p. 59]): if $A \in \Sigma$ and $\mu(A) > 0$ then there exists $B \in \Sigma$, $B \subset A$, such that $0 < \mu(B) < \infty$.

The equivalent *gauge* (or *Luxemburg*) *norm* of a function $f \in L^\Phi$ is defined by the formula

$$\|f\|_{(\Phi)} = \|f\|_{L^{(\Phi)}(\Omega)} = \inf\left\{k > 0 \,:\, \rho_\Phi\left(\frac{f}{k}\right) \le 1\right\}.$$

This is a norm without any constraint on the measure μ (see [20, p. 54, Theorem 3]).

We will need the following familiar assertion (see [20, item (ii), p. 57]):

Lemma 1.5. *Let*

$$0 \le f_1 \le f_2 \le \dots \le f_m \le \dots$$

be an increasing sequence of nonnegative measurable functions in the Orlicz space $L^{\Phi}(\Omega)$ *(*(Ω,Σ,μ) *is a measure space) and let* $f_m \to f$ *a.e. Then* $\lim_{m\to\infty}\|f_m\|_{(\Phi)} \le \|f\|_{(\Phi)} \le \infty$.

2. Orlicz spaces of differential forms. Let X be a Riemannian manifold of dimension n. Given $x \in X$, denote by $(\omega(x),\theta(x))$ the scalar product of exterior k-forms $\omega(x)$ and $\theta(x)$ on T_xX. This gives a function $x \mapsto (\omega(x),\theta(x))$ on X.

Let $\Phi:\mathbb{R}\to\mathbb{R}$ and $\Psi:\mathbb{R}\to\mathbb{R}$ be two complementary N-functions. Denote by $\tilde{L}^{\Phi}(X,\Lambda^k)$ the class of all measurable k-forms ω such that

$$\rho_\Phi(\omega) := \int_X \Phi(|\omega(x)|)d\mu_X < \infty.$$

Here $d\mu_X$ stands for the volume element of the Riemannian manifold X. We will identify k-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold X, introduce the space $L^{\Phi}(X,\Lambda^k)$ as the class of all measurable k-forms ω satisfying the condition

$$\rho_\Phi(\alpha\omega) < \infty \text{ for some } \alpha > 0.$$

The corresponding Morse–Transue space $M^{\Phi}(X,\Lambda^k)$ is defined as the class of all measurable k-forms ω such that

$$\rho_\Phi(\alpha\omega) < \infty \text{ for all } \alpha > 0.$$

Obviously, $\tilde{L}^{\Phi}(X,\Lambda^k) \subset L^{\Phi}(X,\Lambda^k)$.

As in the case of Orlicz function spaces, the space $L^{\Phi}(X,\Lambda^k)$ is endowed with two equivalent norms: the *gauge norm*

$$\|\omega\|_{(\Phi)} = \inf\left\{K>0 \,:\, \rho_\Phi\left(\frac{\omega}{K}\right) \le 1\right\}$$

and the *Orlicz norm*

$$\|\omega\|_{\Phi} = \sup\left\{\left|\int_X (\omega(x),\theta(x))\,d\mu_X\right| : \theta \in \tilde{L}^{\Psi}(X,\Lambda^k),\ \rho_\Psi(\theta) \le 1\right\}.$$

As in the case of function spaces, it can be proved that $L^{\Phi}(X,\Lambda^k)$ endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a k-form ω is nothing but the gauge norm of its modulus function $|\omega|$. The same holds for the Orlicz norm

([15, Lemma 2.1]). Moreover, similarly to the case of Orlicz function spaces ([20, Proposition 10, p. 81]), we have

Lemma 2.1. *The Orlicz and gauge norms of a k-form $\omega \in L^\Phi(X,\Lambda^k)$ can be calculated by the formulas*

$$\|\omega\|_\Phi = S_\omega := \sup_{\substack{\theta\in M^\Psi(X,\Lambda^k),\\ \|\theta\|_{(\Psi)}\le 1}} \left|\int_X (\omega(x),\theta(x))d\mu_X\right|$$

and

$$\|\omega\|_{(\Phi)} = T_\omega := \sup_{\substack{\theta\in M^\Psi(X,\Lambda^k),\\ \|\theta\|_{\Psi}\le 1}} \left|\int_X (\omega(x),\theta(x))d\mu_X\right|.$$

Proof. For $\theta \in M^\Psi(X,\Lambda^k)$ with $\|\theta\|_{(\Psi)} \le 1$ we have

$$\left|\int_X (\omega(x),\theta(x))d\mu_X\right| \le \int_X |\omega(x)||\theta(x)|d\mu_X \le$$

$$\le \sup_{\substack{g\in M^\Psi(X),\\ \|g\|_{(\Psi)}\le 1}} \left|\int_X |\omega(x)|g(x)d\mu_X\right| = \|\,|\omega|\,\|_\Phi.$$

The last equality here holds by [20, Proposition 10, p. 81].

Thus,

$$S_\omega = \sup_{\substack{\theta\in M^\Psi(X,\Lambda^k),\\ \|\theta\|_{(\Psi)}\le 1}} \left|\int_X (\omega(x),\theta(x))\,d\mu_X\right| \le \|\,|\omega|\,\|_\Phi.$$

On the other hand, let $(g_m)_{m\in\mathbb{N}}$ be a sequence of functions in $M^\Psi(X)$ with $\|g_m\|_{(\Psi)} \le 1$ such that

$$\left|\int_X |\omega(x)|g_m(x)d\mu_X\right| \to \|\,|\omega|\,\|_\Phi \text{ as } m\to\infty.$$

Since

$$\left|\int_X |\omega(x)|g_m(x)d\mu_X\right| \le \int_X |\omega(x)||g_m(x)|d\mu_X \le \|\,|\omega|\,\|_\Phi,$$

we also have

$$\int_X |\omega(x)||g_m(x)|d\mu_X \to \| \, |\omega| \, \|_\Phi \text{ as } m \to \infty.$$

Consider the sequence $(\theta_m)_{m\in\mathbb{N}}$ of k-forms θ_m defined by

$$\theta_m(x) = \begin{cases} |g_m(x)| \frac{\omega(x)}{|\omega(x)|} & \text{if } \omega(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\theta_m\|_{(\Psi)} = \|g_m\| \leq 1$ and

$$\left| \int_X (\omega(x), \theta_m(x)) d\mu_X \right| = \left| \int_X |\omega(x)||g_m(x)|d\mu_X \right| \to \| \, |\omega| \, \|_\Phi$$

as $m \to \infty$. Therefore,

$$\| \, |\omega| \, \|_\Phi \leq \sup_{\substack{\theta \in M^\Psi(X,\Lambda^k), \\ \|\theta\|_{(\Psi)} \leq 1}} \left| \int_X (\omega(x), \theta(x)) d\mu_X \right| = \|\omega\|_\Phi.$$

Thus, we get the desired equality for the Orlicz norm.

For the gauge norm, the equality $\|\omega\|_{(\Phi)} = \| |\omega| \|_{(\Phi)}$ is obvious, and one must only prove that

$$T_\omega = \| |\omega| \|_{(\Phi)},$$

which is done in the same manner as for the Orlicz norm with the use of [20, Proposition 10, p. 81]. □

Below, when this does not lead to confusion, we use the abbreviations

$$L^\Phi = (L^\Phi, \| \cdot \|_\Phi), \quad L^{(\Phi)} = (L^\Phi, \| \cdot \|_{(\Phi)});$$

$$M^\Phi = (M^\Phi, \| \cdot \|_\Phi), \quad M^{(\Phi)} = (M^\Phi, \| \cdot \|_{(\Phi)}).$$

3. The Riesz theorem. Let X be an oriented n-dimensional Riemannian manifold.

For a k-form ω on X, let $*\omega$ be the Hodge dual of ω (an $(n-k)$-form).

The bilinear function

$$\langle \omega, \theta \rangle = \int_X \omega \wedge \theta \tag{1}$$

defines a pairing between $L^\Phi(X, \Lambda^k)$ and $L^{(\Psi)}(X, \Lambda^k)$ (and between $L^{(\Phi)}(X, \Lambda^k)$ and $L^\Psi(X, \Lambda^k)$). The integral on the right-hand side of (1) exists because

$$\omega \wedge \theta = (-1)^{kn-k}(\omega, *\theta) d\mu_X,$$

$$|(\omega, *\theta)_X| \le |\omega|_X| * \theta|_X = |\omega|_X |\theta|_X.$$

Hence, we obtain two versions of the Hölder inequality:

$$|\langle \omega, \theta \rangle| \le \|\omega\|_\Phi \|\theta\|_{(\Psi)} \tag{2}$$

and

$$|\langle \omega, \theta \rangle| \le \|\omega\|_{(\Phi)} \|\theta\|_\Psi. \tag{3}$$

Assign to each form $\theta \in L^{(\Psi)}(X, \Lambda^{n-k})$ the functional

$$F_\theta(\omega) = \int_X \omega \wedge \theta. \tag{4}$$

By (2) and (3), we have

$$|F_\theta(\omega)| \le \|\omega\|_\Phi \|\theta\|_{(\Psi)}; \quad |F_\theta(\omega)| \le \|\omega\|_{(\Phi)} \|\theta\|_\Psi. \tag{5}$$

Theorem 3.1. *If Φ is an N-function then the correspondence $\theta \mapsto F_\theta$ yields isometric isomorphisms*

$$L^{(\Psi)}(X, \Lambda^{n-k}) \overset{\cong}{\to} (M^\Phi(X, \Lambda^k))'; \quad L^\Psi(X, \Lambda^{n-k}) \overset{\cong}{\to} (M^{(\Phi)}(X, \Lambda^k))'.$$

Proof. Let us prove the first isomorphism.

By (5), $\|F_\theta\| \le \|\theta\|_{(\Psi)}$. Show that an arbitrary continuous functional $F \in (M^\Phi(X, \Lambda^k))'$ is representable uniquely in the form (4). Let $h : V \to \mathbb{R}^n$, $V \subset X$ be a local chart of X and let U be an open set with compact closure $\operatorname{cl}_X U \subset V$; then U is endowed with two metrics: the metric ρ of the Riemannian manifold X and the metric $\bar{\rho}$ induced by h from the standard metric on $\mathbb{R}^n$. It is not hard to see that the L^Φ-spaces (M^Φ-spaces) of k-forms on U $L^\Phi(U, \Lambda^k, \rho)$ and $L^{(\Phi)}(U, \Lambda^k, \rho)$

($M^\Phi(U,\Lambda^k,\rho)$ and $M^{(\Phi)}(U,\Lambda^k,\rho)$) corresponding to these metrics coincide and have equivalent norms. Making use of the Riesz theorem on the general form of a linear functional on the function space M^Φ, we, involving the coordinate representation of differential forms, conclude that every functional $f \in (M^\Phi(U,\Lambda^k,\bar\rho))'$ is uniquely representable in the form

$$f(\alpha) = \int\limits_X \alpha \wedge \theta_f, \quad \theta_f \in L^{(\Psi)}(U,\Lambda^{n-k},\bar\rho).$$

By the equivalence of the norms in $M^\Phi(U,\Lambda^k,\rho)$ and $M^\Phi(U,\Lambda^k,\bar\rho)$, the same holds for functionals in $M^\Phi(U,\Lambda^k,\rho)$. Therefore, for $F \in (M^\Phi(X,\Lambda^k))'$ and an open set U with compact closure, there is a unique form $\theta_U \in L^{(\Psi)}(U,\Lambda^{n-k})$ such that

$$F(\omega) = \int\limits_U \omega \wedge \theta_U \quad \text{for every } \omega \in M^\Phi(U,\Lambda^k).$$

Given two sets U_1 and U_2 as above, the forms θ_{U_1} and θ_{U_2} coincide on $U_1 \cap U_2$ by the uniqueness of $\theta_{U_1\cap U_2}$. Thus, all forms θ_U defined for different U agree with each other and thus define an $(n-k)$-form θ on X. The form θ belongs to $L^{(\Psi)}(X,\Lambda^{n-k})$ locally, satisfies the condition

$$F(\omega) = \int\limits_X \omega \wedge \theta \quad \text{for all } \omega \in M^\Phi(X,\Lambda^k) \text{ with compact support,}$$

and is defined by this condition uniquely.

Consider a compact set $Y \subset X$. Let $g \in M^\Phi(X)$ be a function with compact support contained in Y having $\|g\|_\Phi \le 1$. Let β_g be the k-form on X defined by the formula

$$\beta_g(x) = \begin{cases} (-1)^{k(n-k)} \frac{g(x)}{|\theta(x)|}(*\theta(x)) & \text{if } x \in Y \text{ and } \theta(x) \ne 0; \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$F(\beta_g) = \int\limits_Y \beta_g \wedge \theta = (-1)^{k(n-k)} \int\limits_Y \frac{g(x)}{|\theta(x)|}(*\theta(x)) \wedge \theta(x) = \int\limits_Y g(x)|\theta(x)|\,d\mu_X.$$

Since $\|g\|_\Phi \le 1$, this gives

$$\left| \int\limits_Y g(x)|\theta(x)|d\mu_X \right| = |F(\beta_g)| \le \|F\|.$$

Hence, using Lemma 2.1, we obtain

$$\|\theta|_Y\|_{(\Psi)} = \|\, |\theta|_Y|\, \|_{(\Psi)} = \sup_{g\in M^\Phi(Y);\ \|g\|_\Phi \le 1} \left| \int\limits_Y g(x)|\theta(x)|d\mu_X \right| \le \|F\|.$$

Let $Y_1 \subset Y_2 \subset \cdots \subset Y_m \subset \cdots \subset X$ be an exhaustion of X by compact sets and let θ_m be the restriction of θ to Y_m. Put $f_m = |\theta_m|$. Then the sequence $\{f_m\}_{m\in\mathbb{N}}$ satisfies the conditions of Lemma 1.5. Since $\|f_m\|_{(\Psi)} \le \|F\|$, the function $\lim\limits_{m\to\infty} f_m = |\theta|$ lies in $L^{(\Psi)}(X)$, and so $\theta \in$ $\in L^{(\Psi)}(X, \Lambda^{n-k})$ and

$$\|\theta\|_{(\Psi)} = \lim_{m\to\infty} \|\theta_m\|_{(\Psi)} \le \|F\|. \tag{6}$$

The functionals F and F_θ coincide on the set of forms in $M^\Phi(X, \Lambda^k)$ having compact support, which is, as in the case of Orlicz function spaces, dense in $M^\Phi(X, \Lambda^k)$. Thus,

$$F(\omega) = \omega \wedge \theta$$

for all $\omega \in M^\Phi(X, \Lambda^k)$. Combining (2) and (6), we infer that $\|F_\theta\| =$ $= \|\theta\|_{(\Psi)}$.

Let us now establish the second isomorphism

$$L^\Psi(X, \Lambda^{n-k}) \overset{\cong}{\to} (M^{(\Phi)}(X, \Lambda^k))'.$$

Let $F \in (M^{(\Phi)}(X, \Lambda^k))'$. Then, as above, we see that there exists a unique $(n-k)$-form θ belonging to L^Ψ locally that satisfies the condition

$$F(\omega) = \int\limits_X \omega \wedge \theta \quad \text{for all } \omega \in M^{(\Phi)}(X, \Lambda^k) \text{ with compact support.}$$

Using Lemma 2.1, we verify in the same manner as for $\|\cdot\|_\Psi$ that, given any compact set $Y \subset X$,

$$\|\theta|_Y\|_\Psi \le \|F\|.$$

Because of the inequalities

$$\|\cdot\|_{(\Psi)} \leq \|\cdot\|_{\Psi} \leq 2\|\cdot\|_{(\Psi)},$$

we have

$$\|\theta|_Y\|_{(\Psi)} \leq \|F\|.$$

Taking an exhaustion $Y_1 \subset Y_2 \subset \cdots \subset Y_m \subset \cdots \subset X$ of X by compact sets, we as above conclude that $\theta \in L^{\Psi}$.

Now, the functionals F and F_θ coincide on the dense set of forms with compact support in $M^{(\Phi)}(X, \Lambda^k)$ and hence on $M^{(\Phi)}(X, \Lambda^k)$. By Lemma 2.1,

$$\|F\| = \|F_\theta\| = \sup_{\substack{\theta \in M^{\Psi}(X,\Lambda^k), \\ \|\theta\|_{(\Phi)} \leq 1}} \left| \int_X \omega \wedge \theta \right| = \|\theta\|_{\Phi}.$$

The theorem is completely proved. □

4. The dual spaces to L^{Φ}-related spaces of differential forms. Throughout this section, X is an oriented smooth Riemannian manifold of dimension n and (Φ_1, Ψ_1) and (Φ_2, Ψ_2) are pairs of conjugate N-functions.

Introduce some spaces of differential forms.

For $A \in \{L, M\}$ and $\langle \Phi_i \rangle \in \{\Phi_i, (\Phi_i)\}$, denote by $A^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)$ the space $A^{\Phi_1}(X, \Lambda^k) \oplus A^{\Phi_2}(X, \Lambda^{k+1})$ with the norm

$$\|(\alpha, \beta)\|_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle} = \|\alpha\|_{\langle \Phi_1 \rangle} + \|\beta\|_{\langle \Phi_2 \rangle}.$$

Given $(\alpha, \beta) \in M^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)$ and $(\omega, \theta) \in L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X)$, where

$$\overline{\langle \Psi_i \rangle} = \begin{cases} (\Psi_i) & \text{if } \langle \Phi_i \rangle = \Phi_i, \\ \Psi_i & \text{if } \langle \Phi_i \rangle = (\Phi_i), \end{cases}$$

we put

$$\langle (\alpha, \beta), (\omega, \theta) \rangle = (-1)^k \langle \alpha, \theta \rangle + \langle \beta, \omega \rangle. \tag{7}$$

Theorem 3.1 implies that the pairing (7) defines an isometric isomorphism

$$(M^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X))' \cong L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X).$$

Moreover,

$$|\langle (\alpha, \beta), (\omega, \theta) \rangle| \leq \|(\alpha, \beta)\|_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle} \cdot \|(\omega, \theta)\|_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}.$$

A differential $(k+1)$-form $\theta \in L^1_{\mathrm{loc}}(X, \Lambda^{k+1})$ on X is called *the weak exterior differential* (or *derivative*) of a k-form $\omega \in L^1_{\mathrm{loc}}(X, \Lambda^k)$ (which is written as $d\omega = \theta$) if,

$$\int\limits_X \theta \wedge u = (-1)^{k+1} \int\limits_X \omega \wedge du$$

for any $u \in \mathcal{D}^{n-k-1}(X)$, where $\mathcal{D}^l(X)$ is the set of smooth l-forms on X with compact support included in $\mathrm{Int}\, X$.

Let Φ_1 and Φ_2 be N-functions. For $0 \le k \le n$, put

$$\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X) = \left\{ \omega \in L^{\langle\Phi_1\rangle}(X, \Lambda^k) \,:\, d\omega \in L^{\langle\Phi_2\rangle}(X, \Lambda^{k+1}) \right\}.$$

This is a Banach space with the norm

$$\|\omega\|_{\langle\Phi_1\rangle,\langle\Phi_2\rangle} = \|\omega\|_{\langle\Phi_1\rangle} + \|d\omega\|_{\langle\Phi_2\rangle}.$$

From now on we assume that $\Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2$, and hence also $\Psi_1, \Psi_2 \in \Delta_2 \cap \nabla_2$.

If $\Phi \in \Delta_2 \cap \nabla_2$ then, as is well known, the spaces L^Φ and M^Φ coincide and hence, by Theorem 3.1, the space L^Φ is reflexive. Thus, there is no need in the spaces $M^*_{*,*}$. We will often assume that the space $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ is embedded in $L^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ by identifying a form $\alpha \in \Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ with the pair $(\alpha, d\alpha) \in L^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$.

Given a subspace $H \subset L^k_{\langle\Phi_1,\Phi_2\rangle}$, denote by $H^\perp$ the annihilator of H in $L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$ with respect to the pairing (7). Since this pairing satisfies

$$\langle(\alpha,\beta),(\omega,\theta)\rangle = (-1)^{k(n-k-1)}\langle(\omega,\theta),(\alpha,\beta)\rangle,$$

there is no difference between the pairings between $L^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ and $L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$ and between $L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$ and $L^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$.

The definition of $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ implies that

$$\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X) = (\mathcal{D}^{n-k-1}(X))^\perp.$$

Put $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X) = (\Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X))^\perp$. Since $\mathcal{D}^{n-k-1}(X) \subset \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$, we have $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X) \subset \Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$.

Observe that if $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)=\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ then $\Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle,\langle\Psi_1\rangle},0}(X)=$ $=\Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle,\langle\Psi_1\rangle}}(X)$.

Lemma 4.1. *The following hold for $\Phi_1,\Phi_2\in\Delta_2\cap\nabla_2$:*

(1) Smooth forms constitute a dense set in $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$.

(2) Smooth forms with compact support constitute a dense set in $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$.

Proof. Item (1) stems from the only theorem of [15] about the properties of the de Rham regularization operators in Orlicz spaces of differential forms. Prove (2). Denote the closure of $\mathcal{D}^k(X)$ in $L_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ by $\overline{\mathcal{D}^k(X)}$. Then, by [21, Theorem 4.7],

$$\overline{\mathcal{D}^k(X)}=((\mathcal{D}^k)^\perp)^\perp=\left(\Omega^k_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)\right)^\perp=\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X).$$

□

Lemma 4.2. *If $\Phi_1,\Phi_2\in\Delta_2\cap\nabla_2$ and a form $\omega\in\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ has compact support then $\omega\in\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$.*

Proof. Suppose that $\omega\in\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ has compact support. Assume first that θ is a smooth $(n-k-1)$-form. By Lemma 4.1, there exists a sequence $\{\omega_j\}$ of smooth forms with compact support such that $\omega_j\to\omega$ in norm as $j\to\infty$. Then

$$\langle(\omega,d\omega),(\theta,d\omega)\rangle=\lim_{j\to\infty}\langle(\omega_j,d\omega_j),(\theta,d\theta)\rangle=$$
$$=\lim_{j\to\infty}\int\limits_X\left[(-1)^k\omega_j\wedge d\theta+d\omega_j\wedge\theta\right]=\lim_{j\to\infty}d(\omega_j\wedge\theta)=0. \quad (8)$$

The last equality in (8) is due to the Stokes theorem. Now, let θ be an arbitrary form in $\Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle,\langle\Psi_1\rangle}}(X)$. By Lemma 4.1, there is a sequence $\{\theta_j\}$ of smooth forms converging to θ in norm as $j\to\infty$. Then

$$\langle(\omega,d\omega),(\theta,d\omega)\rangle=\lim_{j\to\infty}\langle(\omega,d\omega),(\theta_j,d\theta_j)\rangle=0.$$

Thus, $\theta\in\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$. □

Each pair of forms $(\omega,\theta)\in L^{n-k}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$ defines by (7) a continuous linear functional on $L^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ and hence on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ and

$\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$. On the last two spaces, this functional is defined by the formula

$$F(\alpha) = \int\limits_X [(-1)^k \alpha \wedge \theta + d\alpha \wedge \omega]. \tag{9}$$

Theorem 4.3. *If $\Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2$ and Ψ_1, Ψ_2 are the corresponding complementary functions then any continuous linear functional on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ (on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$) can be represented in the form (9). A pair of forms (ω, θ) defines the zero functional on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ (on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$) if and only if $\omega \in \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle},0}(X)$ and $\theta = d\omega$ ($\omega \in$ $\in \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)$ and $\theta = d\omega$). The norm of the functional (9) on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ (on $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)$) has the form*

$$\|F\| = \inf\left\{\|\theta + d\beta\|_{\overline{\langle\Psi_1\rangle}} + \|\omega + \beta\|_{\overline{\langle\Psi_2\rangle}} \ : \ \beta \in \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle},0}(X)\right\}$$
$$\left(\|F\| = \inf\left\{\|\theta + d\beta\|_{\overline{\langle\Psi_1\rangle}} + \|\omega + \beta\|_{\overline{\langle\Psi_2\rangle}} \ : \ \beta \in \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)\right\} \right).$$

Proof. In accordance with [21, Theorem 4.9], if H is a closed subspace in a Banach space Y then $Y'/H^\perp = H'$, where the isomorphism is induced by the canonical pairing between Y and Y'. Therefore,

$$\left(\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)\right)' = L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X) \Big/ \left(\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)\right)^\perp =$$
$$= L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle},0}(X)\ ;$$

$$\left(\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)\right)' = L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X) \Big/ \left(\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle,0}(X)\right)^\perp =$$
$$= L^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle\Psi_2\rangle},\overline{\langle\Psi_1\rangle}}(X)\ .$$

□

Theorem 4.4. *If $\Phi_1, \Phi_2 \in \Delta_2 \cap \nabla_2$ and Ψ_1, Ψ_2 are their complementary N-functions then the dual of the space $\Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ is isomorphic to the completion of $\mathcal{D}^{n-k}(X)$ with respect to the norm*

$$\|\omega\| = \inf\left\{\|\omega + d\theta\|_{\overline{\langle\Psi_1\rangle}} + \|\theta\|_{\overline{\langle\Psi_2\rangle}} \ : \ \theta \in \mathcal{D}^{n-k-1}(X)\right\}. \tag{10}$$

This isomorphism is given by the action

$$\langle \alpha, \omega \rangle = (-1)^k \int_X \alpha \wedge \omega. \tag{11}$$

Proof. Consider the embedding $j : L^{\overline{\langle \Psi_1 \rangle}}(X, \Lambda^{n-k}) \to L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X)$ defined by $j(\omega) = (0, \omega)$. Let

$$\pi : L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \to L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$$

be the canonical projection. It is not hard to see that $\pi \circ j$ is a monomorphism. Since the set $S = \{(\omega, \theta) : \omega \in \mathcal{D}^{n-k-1}(X), \theta \in \mathcal{D}^{n-k}(X)\}$ is dense in $L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X)$, $\pi(S)$ is dense in $L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$. Let $\omega \in \mathcal{D}^{n-k-1}(X)$, $\theta \in \mathcal{D}^{n-k}(X)$. Since $(\omega, d\omega) \in \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$, we have $\pi(\omega, \theta) = \pi(0, \theta - d\omega) = \pi \circ j(\theta - d\omega)$. Hence, the set $\pi \circ j(\mathcal{D}^{n-k})$ is dense in $L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$. Moreover,

$$\|\pi \circ j(\omega)\|_{L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)} = \\ = \inf \left\{ \|\omega + d\theta\|_{\overline{\langle \Psi_1 \rangle}} + \|\theta\|_{\overline{\langle \Psi_2 \rangle}} \ : \ \theta \in \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X) \right\}.$$

By Lemma 4.1(2), the set $\mathcal{D}^{n-k-1}(X)$ is dense in $\Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$. Hence,

$$\|\pi \circ j(\omega)\|_{L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)} = \\ = \inf \left\{ \|\omega + d\theta\|_{\overline{\langle \Psi_1 \rangle}} + \|\theta\|_{\overline{\langle \Psi_2 \rangle}} \ : \ \theta \in \mathcal{D}^{n-k-1}(X) \right\}.$$

Thus, the space $L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X)$ is isomorphic to the completion of $\mathcal{D}^{n-k}(X)$ with respect to the norm (10). Now, in view of [21, Theorem 4.9], if H is a closed subspace in a Banach space Y then $(Y/H)' = = H^{\perp}$, where the isomorphism is induced by the canonical pairing between Y and Y'. Thus, $\left(L^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}}(X) \Big/ \Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X) \right)' = = \left(\Omega^{n-k-1}_{\overline{\langle \Psi_2 \rangle}, \overline{\langle \Psi_1 \rangle}, 0}(X) \right)^{\perp} = \Omega^k_{\langle \Phi_1 \rangle, \langle \Phi_2 \rangle}(X)$, and the first claim of the theorem is established.

Further, since

$$\langle(\alpha, d\alpha), (0, \omega)\rangle = (-1)^k \int_X \alpha \wedge \omega,$$

the form $\alpha \in \Omega^k_{\langle\Phi_1\rangle,\langle\Phi_2\rangle}(X)$ acts at the forms $\pi \circ j(\omega)$, $\omega \in \mathcal{D}^{n-k}(X)$, by the formula

$$\langle \alpha, \pi \circ j(\omega)\rangle = (-1)^k \int_X \alpha \wedge \omega.$$

The theorem is proved. □

5. Hölder–Poincaré duality for $L_{\Phi_I,\Phi_{II}}$-cohomology. Let X be an oriented Riemannian manifold of dimension n.

Given N-functions Φ_I and Φ_{II}, consider the spaces

$$Z^k_{\langle\Phi_{II}\rangle}(X) = \{\omega \in L^{\langle\Phi_{II}\rangle}(X, \Lambda^k) \,:\, d\omega = 0\};$$

$$B^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X) = \{\omega \in L^{\langle\Phi_{II}\rangle}(X, \Lambda^k) \,:\, \omega = d\beta \text{ for some } \beta \in L^{\langle\Phi_I\rangle}(X, \Lambda^{k-1})\}.$$

Denote by $\overline{B}^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X)$ the closure of $B^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X)$ in $L^{\langle\Phi_{II}\rangle}(X, \Lambda^k)$. The quotient spaces

$$H^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X) := Z^k_{\langle\Phi_{II}\rangle}(X)/B^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X)$$

and

$$\overline{H}^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X) := Z^k_{\langle\Phi_{II}\rangle}(X)/\overline{B}^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}(X)$$

are called the kth $L_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}$*-cohomology* and the kth *reduced* $L_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}$*-cohomology* of the Riemannian manifold X, the latter cohomology being a Banach space.

If $\Phi_I = \Phi_{II} = \Phi$ then we use the notations $\Omega^k_{\langle\Phi\rangle}(X)$, $H^k_{\langle\Phi\rangle}(X)$, and $\overline{H}^k_{\langle\Phi\rangle}(X)$ instead of $\Omega^k_{\langle\Phi\rangle,\langle\Phi\rangle}(X)$, $H^k_{\langle\Phi\rangle,\langle\Phi\rangle}(X)$, and $\overline{H}^k_{\langle\Phi\rangle,\langle\Phi\rangle}(X)$ respectively. Thus, the $L_{\langle\Phi\rangle}$*-cohomology* $H^k_{\langle\Phi\rangle}(X)$ (respectively, the *reduced* $L_{\langle\Phi\rangle}$*-cohomology* $\overline{H}^k_{\langle\Phi\rangle}(X)$) is the kth cohomology (respectively, the kth reduced cohomology) of the cochain complex $\{\Omega^*_{\langle\Phi\rangle}(X), d\}$.

The *kth interior reduced* $L_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}$*-cohomology* of a Riemannian manifold X is the Banach space

$$\overline{H}^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle,0}(X) = Z^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle,0}(X) \Big/ \overline{d\mathcal{D}^{k-1}(X)}\,,$$

where $\overline{d\mathcal{D}^{k-1}(X)}$ is the closure of $d\mathcal{D}^k(X)$ in $L^{\langle\Phi_{II}\rangle}(X,\Lambda^k)$ and

$$Z^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle,0}(X) = \operatorname{Ker}\left\{d : \Omega^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle} \to \Omega^{k+1}_{\langle\Phi_{II}\rangle,\langle\Phi_{II}\rangle}\right\} \cap \overline{\mathcal{D}^k(X)}^{\Omega^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}}.$$

Thus, a k-form θ belongs to $Z^k_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle,0}(X)$ if and only if $\theta \in$ $\in L^{\langle\Phi_I\rangle}(X,\Lambda^k)$, $d\theta = 0$, and there is a sequence is a weakly closed forms $\theta_j \in \mathcal{D}^k(X)$ such that

$$\|\theta_j - \theta\|_{\langle\Phi_I\rangle} \to 0 \text{ and } \|d\theta_j\|_{\langle\Phi_{II}\rangle} \to 0 \text{ as } j \to \infty.$$

The quotient (semi)norm on each of the above-introduced cohomology spaces depends on the choice of the norm on L^{Φ_I} and $L^{\Phi_{II}}$ but the resulting topology does not.

From now on, we assume all N-functions under consideration to belong to $\Delta_2 \cap \nabla_2$.

In [11], Gol′dshtein and Troyanov realized the kth $L_{q,p}$-cohomology as the kth cohomology of some Banach complex. Here we apply this approach to $L_{\langle\Phi_I\rangle,\langle\Phi_{II}\rangle}$-cohomology.

Fix an $(n+1)$-tuple of N-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ and put

$$\Omega^k_{\mathcal{F}}(X) = \Omega^k_{\Phi_k,\Phi_{k+1}}(X); \quad \Omega^k_{(\mathcal{F})}(X) = \Omega^k_{(\Phi_k),(\Phi_{k+1})}(X).$$

Use the unified notation $\Omega^k_{\langle\mathcal{F}\rangle}(X)$ for $\Omega^k_{\mathcal{F}}(X)$ and $\Omega^k_{(\mathcal{F})}(X)$. Since the weak exterior differential is a bounded operator $d : \Omega^k_{\langle\mathcal{F}\rangle}(X) \to \Omega^{k+1}_{\langle\mathcal{F}\rangle}(X)$, we obtain a Banach complex

$$0 \to \Omega^0_{\langle\mathcal{F}\rangle}(X) \to \Omega^1_{\langle\mathcal{F}\rangle}(X) \to \dots \to \Omega^k_{\langle\mathcal{F}\rangle}(X) \to \dots \to \Omega^n_{\langle\mathcal{F}\rangle}(X) \to 0.$$

The $L_{\langle\mathcal{F}\rangle}$*-cohomology* $H^k_{\langle\mathcal{F}\rangle}(X)$ (respectively, the *reduced* $L_{\langle\mathcal{F}\rangle}$*-cohomology* $\overline{H}^k_{\langle\mathcal{F}\rangle}(X)$) of X is the kth cohomology (respectively, the kth reduced cohomology) of the Banach complex $(\Omega^*_{\langle\mathcal{F}\rangle}, d)$.

The above-defined cohomology spaces $H^k_{\langle\mathcal{F}\rangle}(X)$ and $\overline{H}^k_{\langle\mathcal{F}\rangle}(X)$ in fact depend only on Φ_{k-1} and Φ_k:

$$H^k_{\langle\mathcal{F}\rangle}(X) = H^k_{\langle\Phi_{k-1}\rangle,\langle\Phi_k\rangle}(X) = Z^k_{\langle\Phi_k\rangle}(X) \Big/ B^k_{\langle\Phi_{k-1}\rangle,\langle\Phi_k\rangle}\ ;$$
$$\overline{H}^k_{\langle\mathcal{F}\rangle}(X) = \overline{H}^k_{\langle\Phi_{k-1}\rangle,\langle\Phi_k\rangle}(X) = Z^k_{\langle\Phi_k\rangle}(X) \Big/ \overline{B}^k_{\langle\Phi_{k-1}\rangle,\langle\Phi_k\rangle}\ .$$

Denote by $\Omega^k_{\langle\mathcal{F}\rangle,0}(X)$ the closure of $\mathcal{D}^k(X)$ in $\Omega^k_{\langle\mathcal{F}\rangle}(X)$. The *interior reduced $L_{\langle\mathcal{F}\rangle}$-cohomology* of X is the reduced cohomology of the Banach complex

$$0 \to \Omega^0_{\langle\mathcal{F}\rangle,0}(X) \to \Omega^1_{\langle\mathcal{F}\rangle,0}(X) \to \cdots \to \Omega^k_{\langle\mathcal{F}\rangle,0}(X) \to \cdots \to \Omega^n_{\langle\mathcal{F}\rangle,0}(X) \to 0;$$

$$\overline{H}^k_{\langle\mathcal{F}\rangle,0}(X) = \overline{H}^k_{\langle\Phi_k\rangle,\langle\Phi_{k+1}\rangle,0}(X) = Z^k_{\langle\Phi_k\rangle,\langle\Phi_{k+1}\rangle,0}(X) \Big/ \overline{d\mathcal{D}^{k-1}(X)}^{L^{\Phi_k}(X,\Lambda^k)}.$$

The dual of an $(n+1)$-tuple of N-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ is the $(n+1)$-tuple $\mathcal{F}' = \{\Psi_0, \Psi_1, \dots, \Psi_n\}$, where Ψ_k and Φ_{n-k} are complementary N-functions for all k. Henceforth, we assume all N-functions to belong to the class $\Delta_2 \cap \nabla_2$.

Fix an $(n+1)$-tuple of N-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ and let $\mathcal{F}' = \{\Psi_0, \Psi_1, \dots, \Psi_n\}$ be its dual $(n+1)$-tuple. For $-1 \le k \le n$, introduce the vector spaces

$$\mathcal{P}^k_{\langle\mathcal{F}\rangle}(X) = L^k_{\langle\Phi_k\rangle,\langle\Phi_{k+1}\rangle}(X) = L^{\langle\Phi_k\rangle}(X,\Lambda^k) \oplus L^{\langle\Phi_{k+1}\rangle}(X,\Lambda^{k+1})$$

(here $L^{\langle\Phi_k\rangle}(X,\Lambda^k) = 0$ for $k = -1, n+1$). If $(\alpha,\beta) \in \mathcal{P}_{\langle\mathcal{F}\rangle}(X)$ with $\alpha \in L^{\langle\Phi_k\rangle}(X,\Lambda^k)$ and $\beta \in L^{\langle\Phi_{k+1}\rangle}(X,\Lambda^{k+1})$ then $\mathcal{P}_{\langle\mathcal{F}\rangle}(X)$ is endowed with the norm

$$\|(\alpha,\beta)\|_{\mathcal{P}_{\langle\mathcal{F}\rangle}(X)} = \|\alpha\|_{\langle\Phi_k\rangle} + \|\beta\|_{\langle\Phi_{k+1}\rangle}.$$

Let $d_{\mathcal{P}} : \mathcal{P}^k_{\langle\mathcal{F}\rangle}(X) \to \mathcal{P}^{k+1}_{\langle\mathcal{F}\rangle}(X)$ be defined as

$$d_{\mathcal{P}}(\alpha,\beta) = (\beta, 0).$$

The so-obtained Banach complex $\left(\mathcal{P}^*_{\langle\mathcal{F}\rangle}(X), d_{\mathcal{P}}\right)$ has trivial cohomology.

Lemma 5.1. *Let $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ be an $(n+1)$-tuple of N-functions and let $\mathcal{F}' = \{\Psi_0, \Psi_1, \dots, \Psi_n\}$ be its dual $(n+1)$-tuple. Then the spaces*

$\mathcal{P}^k_{\langle\mathcal{F}\rangle}(X)$ *and* $\mathcal{P}^{n-k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *(here, as above, the bar changes the type of the norm) are dual with respect to the pairing*

$$\langle(\alpha,\beta),(\omega,\theta)\rangle = \int\limits_X \left((-1)^k\alpha\wedge\omega + \beta\wedge\theta\right). \tag{12}$$

Lemma 5.1 easily follows from Theorem 4.3.

Lemma 5.2. *The operators*

$$d : \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X) \to \mathcal{P}^{k}_{\overline{\langle\mathcal{F}'\rangle}}(X) \ \ \textit{and} \ \ d : \mathcal{P}^{n-k-1}_{\langle\mathcal{F}\rangle}(X) \to \mathcal{P}^{n-k}_{\langle\mathcal{F}\rangle}(X)$$

are adjoint.

Proof. If $(\alpha,\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ and $(\omega,\theta)\in\mathcal{P}^{n-k-1}_{\langle\mathcal{F}\rangle}(X)$ then

$$\langle d(\alpha,\beta),(\omega,\theta)\rangle = \langle(\beta,0),(\omega,\theta)\rangle = \int\limits_X (-1)^k\beta\wedge\theta,$$

$$\langle(\alpha,\beta),d(\omega,\theta)\rangle = \langle(\alpha,\beta),(\theta,0)\rangle = \int\limits_X \beta\wedge\theta. \qquad \square$$

Put

$$\Sigma^k_{\langle\mathcal{F}\rangle}(X) = \left\{(\omega,d\omega)\in\mathcal{P}^k_{\langle\mathcal{F}\rangle}(X) \ : \ \omega\in\Omega^k_{\langle\mathcal{F}\rangle}(X)\right\};$$
$$\Sigma^k_{\langle\mathcal{F}\rangle,0}(X) = \left\{(\omega,d\omega)\in\mathcal{P}^k_{\langle\mathcal{F}\rangle}(X) \ : \ \omega\in\Omega^k_{\langle\mathcal{F}\rangle,0}(X)\right\}.$$

Clearly, these spaces form Banach complexes $\Sigma_{\langle\mathcal{F}\rangle}(X)$ and $\Sigma_{\langle\mathcal{F}\rangle,0}(X)$ which are isomorphic to $\Omega_{\langle\mathcal{F}\rangle}(X)$ and $\Omega_{\langle\mathcal{F}\rangle,0}(X)$ respectively.

Introduce the following quotient complex of $\mathcal{P}_{\overline{\langle\mathcal{F}'\rangle}}(X)$:

$$\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X) = \mathcal{P}^*_{\overline{\langle\mathcal{F}'\rangle}}(X) \Big/ \Sigma^*_{\overline{\langle\mathcal{F}'\rangle,0}}(X).$$

What was said above implies:

Proposition 5.3. *The graded vector space* $\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *possesses the following properties:*

(1) $\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *is a Banach space with respect to the norm*

$$\|(\omega,\theta)\|_{\mathcal{A}} = \inf\left\{\|\omega+\rho\|_{\overline{\langle\Psi_k\rangle}} + \|\theta+d\rho\|_{\overline{\langle\Psi_{k+1}\rangle}}\right\}.$$

(2) $\mathcal{A}^k_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *is dual to* $\Sigma^{n-k-1}_{\langle\mathcal{F}\rangle}(X)$ *with respect to the pairing (12).*

(3) *The differential* $d_{\mathcal{P}} : \mathcal{P}^k_{\overline{\langle\mathcal{F}'\rangle}}(X) \to \mathcal{P}^{k+1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *induces a differential* $d_{\mathcal{A}} : \mathcal{A}^k_{\overline{\langle\mathcal{F}'\rangle}}(X) \to \mathcal{A}^{k+1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *and* $(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X), d_{\mathcal{A}})$ *is a Banach complex.*

(4) *The operators* $d_{\mathcal{A}} : \mathcal{A}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X) \to \mathcal{A}^k_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *and* $d_{\Sigma} : \Sigma^{n-k-1}_{\langle\mathcal{F}\rangle}(X) \to$ $\to \Sigma^{n-k}_{\langle\mathcal{F}\rangle}(X)$ *are adjoint up to sign with respect to the pairing (12).*

Examine the cohomology of the Banach complex $(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)(X), d_{\mathcal{A}})$. If we put

$$Z^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = \operatorname{Ker} d_{\mathcal{A}} : \mathcal{A}^k_{\overline{\langle\mathcal{F}'\rangle}}(X) \to \mathcal{A}^{k+1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$$

and

$$B^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = \operatorname{Im} d_{\mathcal{A}}\left(\mathcal{A}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$$

and denote by $\overline{B}^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$ the closure of $B^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$ then the cohomology and the reduced cohomology of $\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)$ are the spaces

$$H^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = Z^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)\Big/B^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right);$$
$$\overline{H}^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = Z^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)\Big/\overline{B}^k\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right).$$

We will need the following assertion [11, Lemma 3.1]:

Lemma 5.4. Let $I : Y_0 \times Y_1 \to \mathbb{R}$ be a duality between two reflexive Banach spaces. Let B_0, B_1, A_0, A_1 be linear subspaces such that

$$B_0 \subset A_0 = B_1^{\perp} \subset Y_0; \quad B_1 \subset A_1 = B_0^{\perp} \subset Y_1.$$

Then the pairing $\bar{I} : (A_0/\overline{B}_0) \times (A_1/\overline{B}_1) \to \mathbb{R}$ (with the bars standing for closures) is well-defined and induces duality between $A_0/\overline{B}_0$ and $A_1/\overline{B}_1$.

Lemma 5.5. *The pairing (12) induces a pairing between the reduced cohomologies of* $\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)$ *and* $\Sigma^*_{\langle\mathcal{F}\rangle}(X)$.

Proof. We have

$$B^{k-1}(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)) \subset Z^{k-1}(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)) = \left(B^{n-k}(\Sigma^*_{\langle\mathcal{F}\rangle}(X))\right)^{\perp} \subset \mathcal{A}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)),$$

and, similarly,

$$\operatorname{Im} d_{\Sigma}^{n-k-1} \subset \operatorname{Ker} d_{\Sigma}^{n-k} = \left(\operatorname{Im} d_{\mathcal{A}}^{k-2}\right)^{\perp} \subset \Sigma_{\langle\mathcal{F}\rangle}^{n-k}(X),$$

where the equalities are due to the fact that d_{Σ} and $d_{\mathcal{A}}$ are adjoint operators. It remains to apply Lemma 5.4 with $X_0 = \mathcal{A}_{\overline{\langle\mathcal{F}'\rangle}}^{k-1}$ and $X_1 = \Sigma_{\langle\mathcal{F}\rangle}^{n-k}(X)$. □

Lemma 5.6. *The reduced cohomology of the Banach complex $(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle},0}(X), d_{\mathcal{A}})$ is isomorphic to the interior cohomology of X up to a shift:*

$$\overline{H}^{k}_{\overline{\langle\mathcal{F}'\rangle}}(X) \cong \overline{H}^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right).$$

The isomorphism is induced by the mapping $j : Z^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X) \to \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$, $j(\beta) = (0, \beta)$.

Proof. Every element in $\mathcal{A}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ is represented by an element $(\alpha, \beta) \in \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ modulo $\Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$; thus, (α, β) and (α_1, β_1) represent one element in $\mathcal{A}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ if and only if $\alpha - \alpha_1 = \omega$ and $\beta - \beta_1 = d\omega$, where $\omega \in \Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$.

Further, $(\alpha, \beta) \in \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ represents an element of $Z^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$ whenever $d_{\mathcal{P}}(\alpha, \beta) = (\beta, 0) \in \Sigma^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$, that is, $\beta \in Z^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$. Thus,

$$Z^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = \left\{(\alpha, \beta) \in \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X) \ : \ \beta \in Z^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\} \Big/ \Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X).$$

Similarly, (α, β) represents an element in $B^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$ if there is $(\gamma, \delta) \in \mathcal{P}^{k-2}_{\overline{\langle\mathcal{F}'\rangle}}(X)$ with $(\alpha, \beta) = d_{\mathcal{A}}(\gamma, \delta) = (\delta, 0)$ modulo $\Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$, which means that $\beta = d\omega \in B^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$. Thus,

$$B^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = \left\{(\alpha, \beta) \in \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X) \ : \ \beta \in B^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\} \Big/ \Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$$

and

$$\overline{B}^{k-1}\left(\mathcal{A}^{*}_{\overline{\langle\mathcal{F}'\rangle}}(X)\right) = \left\{(\alpha, \beta) \in \mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X) \ : \ \beta \in \overline{B}^{k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\} \Big/ \Sigma^{k-1}_{\overline{\langle\mathcal{F}'\rangle},0}(X).$$

Therefore,

$$H^{k-1}\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)=\frac{\left\{(\alpha,\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \beta\in Z^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}{\left\{(\tilde\alpha,\tilde\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \tilde\beta\in B^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}=$$

$$=\frac{\left\{(0,\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \beta\in Z^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}{\left\{(0,\tilde\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \tilde\beta\in B^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}.$$

Thus, the embedding $j:Z^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\to\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)$, $j(\beta)=(0,\beta)$, induces an algebraic isomorphism $j_*:H^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\overset{\cong}{\to}H^{k-1}\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$. We also have the relation

$$\overline{H}^{k-1}\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)=\frac{\left\{(0,\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \beta\in Z^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}{\left\{(0,\tilde\beta)\in\mathcal{P}^{k-1}_{\overline{\langle\mathcal{F}'\rangle}}(X)\ :\ \tilde\beta\in \overline{B}^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\right\}}.$$

The quotient on the right-hand side is endowed with the natural quotient norm and j induces an isometric isomorphism $\bar{j}_*:\overline{H}^k_{\overline{\langle\mathcal{F}'\rangle},0}(X)\overset{\cong}{\to}$ $\overset{\cong}{\to}\overline{H}^{k-1}\left(\mathcal{A}^*_{\overline{\langle\mathcal{F}'\rangle}}(X)\right)$. □

Thus, we have

Theorem 5.7. *Let X be a smooth n-dimensional oriented Riemannian manifold and let $\mathcal{F}=(\Phi_0,\Phi_1,\dots\Phi_n)$ and $\mathcal{F}'=(\Psi_0,\Psi_1,\dots,\Psi_n)$ be dual sequences of N-functions with $\Phi_i\in\Delta_2\cap\nabla_2$. Then the Banach spaces $\overline{H}^k_{\langle\mathcal{F}\rangle}(X)$ and $\overline{H}^{n-k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$ are dual with respect to the pairing $\langle\omega,\theta\rangle=$ $=\int_X\omega\wedge\theta$ for $\omega\in Z^k_{\langle\mathcal{F}\rangle}(X)$ and $\theta\in Z^{n-k}_{\overline{\langle\mathcal{F}'\rangle},0}(X)$.*

This gives the following duality theorem for $L_{\Phi_I,\Phi_{II}}$-cohomology:

Theorem 5.8. *Let X be an oriented n-dimensional Riemannian manifold. If Φ_I,Φ_{II} are N-functions belonging to $\Delta_2\cap\nabla_2$ and Ψ_I and Ψ_{II} are their respective complementary N-functions then $\overline{H}^k_{\Phi_I,\Phi_{II}}(X)$ is isomorphic to the dual of $\overline{H}^{n-k}_{(\Psi_{II}),(\Psi_I),0}(X)$ and $\overline{H}^k_{(\Phi_I),(\Phi_{II})}(X)$ is isomorphic to the dual of $\overline{H}^{n-k}_{\Psi_{II},\Psi_I,0}(X)$. The dualities are given by the pairing*

$$\langle[\omega],[\theta]\rangle=\int_X\omega\wedge\theta.$$

Proof. The theorem results from Theorem 5.7 by considering any sequence of N-functions $(\Phi_0, \ldots, \Phi_n)$ with $\Phi_{k-1} = \Phi_I$ and $\Phi_k = \Phi_{II}$ and its dual sequence. □

Acknowledgment. The author is indebted to the anonymous referees whose helpful comments and suggestions have improved the exposition of the paper.

References

[1] Agarwal R. P., Ding S., and Nolder C. A. *Inequalities for Differential Forms.* Springer, 2009. DOI: 10.1007/978-0-387-68417-8

[2] Cheeger J., Gromov M. *Bounds on the Von Neumann dimension and the Gauss–Bonnet formula for open manifolds.* J. Diff. Geom., 1985, vol. 21, no. 1, pp. 1–34. DOI: 10.1016/0040-9383(86)90039-X

[3] Cheeger J., Gromov M. *L_2-cohomology and group cohomology.* Topology, 1986, vol. 25, no. 1, pp. 189–215. DOI: 10.1016/0040-9383(86)90039-X

[4] Ding S., Xing Y. *Imbedding theorems in Orlicz-Sobolev space of differential forms.* Nonlinear Anal., 2014, vol. 96, pp. 87–95. DOI: 10.1016/j.na.2013.11.005

[5] Gol′dshtein V. M., Kuz′minov V. I., and Shvedov I. A. *A property of the de Rham regularization operator.* Siberian Math. J., 1984, vol. 25, no. 2, pp. 251–257. DOI: 10.1007/BF00971462

[6] Gol′dshtein V. M., Kuz′minov V. I., and Shvedov I. A. *Dual spaces of spaces of differential forms.* Siberian Math. J., 1986, vol. 27, no. 1, pp. 35–44. DOI: 10.1007/BF00969340

[7] Gol′dshtein V. M., Kuz′minov V. I., and Shvedov I. A. *Reduced L_p-cohomology of warped cylinders.* Siberian Math. J., 1990, vol. 31, no. 5, pp. 716–727. DOI: 10.1007/BF00974484

[8] Gol′dshtein V. M., Kuz′minov V. I., and Shvedov I. A. *L_p-cohomology of warped cylinders.* Siberian Math. J., 1990, vol. 31, no. 6, pp. 919–925. DOI: 10.1007/BF00970057

[9] Gol′dshtein V., Troyanov M. *Sobolev inequalities for differential forms and $L_{q,p}$-cohomology.* J. Geom. Anal., 2006, vol. 16, no. 4, pp. 597–632. DOI: 10.1007/BF02922133

[10] Gol′dshtein V., Troyanov M. *$L_{q,p}$-cohomology of Riemannian manifolds with negative curvature.* In: *International Mathematical Series, Sobolev Spaces in Mathematics II.* Springer New York, 2009, pp. 199–208. DOI: 10.1007/978-0-387-85650-6_9

[11] Gol′dshtein V., Troyanov M. *The Hölder-Poincare duality for* $L_{q,p}$*-cohomology*, Ann. Global Anal. Geom., 2012, vol. 41, no. 1, pp. 25–45. DOI: 10.1007/s10455-011-9269-x

[12] Gromov M. *Asymptotic invariants of infinite groups. Geometric Group Theory, vol. 2.* Cambridge Univ. Press, 1993.

[13] Iwaniec T., Martin G. *Geometric Function Theory and Nonlinear Analysis.* Oxford University Press, 2001.

[14] Johnson C., Ding S. *Integral estimates for the potential operator on differential forms.* Int. J. Anal., 2013, Article ID 108623, 6 p. DOI: 10.1155/2013/108623

[15] Kopylov Ya. A., Panenko R. A. *De Rham regularization operators in Orlicz spaces of differential forms on Riemannian manifolds.* Sib. Élektron. Mat. Izv., 2015, vol. 12, pp. 361–371. DOI: 10.17377/semi.2015.12.030

[16] Kuz′minov V. I., Shvedov I. A. *On compactly-supported approximation of closed differential forms on Riemannian manifolds of special type*. Siberian. Math. J., 1993, vol. 34, no. 3, pp. 486–499. DOI: 10.1007/BF00971224

[17] Kuz′minov V. I., Shvedov I. A. *On the operator of exterior derivation on the Riemannian manifolds with cylindrical ends*. Siberian. Math. J., 2007, vol. 48, no. 3, pp. 500–507. DOI: 10.1007/s11202-007-0052-y

[18] Pansu P. *Cohomologie* L^p *des variétés a courbure negative, cas du degré 1.* Rend. Mat. Univ. Pol. Torino, 1989, Fascicolo Speciale "PDE and Geometry", pp. 95–119.

[19] Pansu P. *Cohomologie* L^p*, espaces homogènes et pincement.* Preprint, Orsay, 1999.

[20] Rao M. M., Ren Z. D. *Theory of Orlicz Spaces.* Pure and Applied Mathematics, vol. 146. Marcel Dekker, 1991.

[21] Rudin W. *Functional Analysis.* McGraw-Hill Book Comp., 1973.

Received May 12, 2017.
In revised form, October 6, 2017.
Accepted August 30, 2017.
Published online October 6, 2017.

Sobolev Institute of Mathematics
4, Akad. Koptyuga st., Novosibirsk 630090, Russia;
Novosibirsk State University
2, Pirogova st., Novosibirsk 630090, Russia
E-mail: yakop@math.nsc.ru