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QUASI-ISOMETRIC MAPPINGS AND THE p-MODULI OF PATH FAMILIES

Abstract. In this article, we study a connection between quasi-isometric mappings of n-dimensional domains and the p-moduli of path families. In particular, we obtain explicit (and sharp) estimates for the distortion of the p-moduli of path families under K-quasi-isometric mappings.

Key words: *p-modulus of path families, p-capacity of the condenser, quasi-isometric mappings*

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1. Introduction. The article is devoted to the study of problems connected with the search for a complete descritpion of quasi-isometric mappings of n-dimensional domains in terms of the p-moduli of families of paths (curves). Note that this problem (for quasi-isometric mappings and also for quasiconformal mappings, space mappings with bounded distortion, mappings with finite distortion, homeomorphisms with finite mean dilatations, mappings with (p, q)-distortion etc) was successfully solved by many mathematicians (see, for example, [1] – [3]; see also [4] – [9]). Our main goal is to obtain explicit (and sharp) estimates for the distortion of the p-moduli of families of paths and curves under K-quasi-isometric mappings. Here we use the following, metric definition of such mappings:

Definition 1. *Let $K \in [1, \infty[$. A homeomorphism $f\colon U_1 \to U_2$ of domains U_1 and U_2 in $\mathbb{R}^n$ is called K-quasi-isometric if*

$$K^{-1} \le \liminf_{y\to x} \frac{|f(y) - f(x)|}{|y - x|} \le \limsup_{y\to x} \frac{|f(y) - f(x)|}{|y - x|} \le K$$

for any $x \in U_1$. A homeomorphism $f\colon U_1 \to U_2$ is called quasi-isometric if it is K-quasi-isometric for some $K \in [1, \infty[$.

Our main result is

Theorem 1. *Suppose that $f\colon U_1 \to U_2$ is a K-quasi-isometric homeomorphism of bounded domains U_1 and U_2 in $\mathbb{R}^n$, where $n \geq 2$ $(1 \leq K < < \infty)$. Then*

$$K^{2-p-n} M_p(\Gamma) \leq M_p(f(\Gamma)) \leq K^{p+n-2} M_p(\Gamma) \qquad (1)$$

for every $p \in]1, \infty[$ and any family Γ of paths γ such that $\operatorname{Im}\gamma \subset \operatorname{cl} U_1$.

Remark 1. *The quantity $M_p(\Gamma)$, where $1 \leq p < \infty$, is called the p-modulus of the path family Γ and defined as*

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{R}(\Gamma)} \int\limits_{\mathbb{R}^n} [\rho(x)]^p dx,$$

where $\mathcal{R}(\Gamma)$ is the set of all nonnegative Borel measurable functions $\rho\colon \mathbb{R}^n \to \dot{\mathbb{R}}$ such that $\int\limits_{\gamma} \rho ds \geq 1$ for every rectifiable path $\gamma \in \Gamma$.

It should be noted that our main result (Theorem 1) is conceptually most close to the results on quasi-isometries in [1].

For example, using Theorem 1 in [10] and our result, Corollary 3 to Theorem 4.4′ in [1], Chapter 5, Section 4, can be supplemented by the following assertion:

Theorem 2. *Under the conditions of Theorem 1,*

$$K^{2-p-n} C_p^1(F_0, F_1; U_1) \leq C_p^1(f(F_0), f(F_1); U_2) \leq K^{p+n-2} C_p^1(F_1, F_2; U_1)$$

for every $p \in]1, \infty[$ and any condenser $(F_0, F_1; U)$.

Remark 2. *$C_p^1(F_0, F_1, U)$ is the p-capacity of the condenser $(F_0, F_1; U)$ (F_0 and F_1 are closed disjoint nonempty sets in $\operatorname{cl} U$, where $U \subset \widetilde{\mathbb{R}}^n = = \mathbb{R}^n \cup \{\infty\}$ is an open set), i.e.,*

$$C_p^1(F_0, F_1; U) = \inf \int\limits_U |\nabla u|^p dx,$$

where infimum is taken over all functions $u \in C^\infty(U) \cap L_p^1(U)$ that are equal to unity (zero) in some neighborhood of F_0 (F_1) (see [11]).

In what follows, for $x \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$, $\operatorname{dist}(x, E) = \inf\limits_{y \in E} |x - y|$, all paths $\gamma\colon [\alpha, \beta] \to \mathbb{R}^n$, where $\alpha, \beta \in \mathbb{R}$, are assumed continuous and non-constant, and $l(\gamma)$ means the length of a path γ.

2. Proof of Theorem 1. The proof of Theorem 1 follows the lines of the proof of the second claim of Theorem 6.5 in [12].

Let Γ be a family of paths in the domain U_1 (i.e., of paths $\gamma\colon [a,b] \to \to \mathbb{R}^n$ such that $\operatorname{Im}\gamma \subset \operatorname{cl} U_1$). Consider the subfamily Γ^* of Γ consisting of all locally rectifiable paths $\gamma \in \Gamma$ such that f is absolutely continuous on every closed subpath of γ. Since f is a quasi-isometry, $f \in ACL_p$ for all $p > 1$ (see, for example, [13, 12], for the definition of the class ACL_p); therefore, $M_p(\Gamma_0) = 0$ for the family Γ_0 of all locally rectifiable paths in U_1 having subpaths on which the mapping f is not absolutely continuous ([13]). The fact that $\Gamma \setminus \Gamma^* \subset \Gamma_0$ and the properties of moduli imply the equality $M_p(\Gamma \setminus \Gamma^*) = 0$. Consequently, $M_p(\Gamma^*) = M_p(\Gamma)$. Therefore, for proving, for example, the left-hand inequality in (1), which we will do below, it suffices to show that $M_p(\Gamma^*) \le K^{p+n-2} M_p(f(\Gamma))$, where $f(\Gamma) = \{f \circ \gamma : \gamma \in \Gamma\}$.

Let E be a Borel subset in U_1 that contains all points $x \in U_1$ at which f is not differentiable and all those points x in U_1 at which f is differentiable but the Jacobian $J(x,f) = 0$, moreover, $\operatorname{mes} E$ $(= \operatorname{mes}_n E) = 0$. Here we use the facts that a quasi-isometric mapping is quasiconformal and the set of points of nondegenerate differentiability of a quasiconformal mapping is a set of full measure with respect to its domain of definition.

Assume that $\widetilde{\rho} \in \mathcal{R}(f(\Gamma^*))$ $(f(\Gamma^*) = \{f \circ \gamma : \gamma \in \Gamma^*\})$, i.e., $\int_{\widetilde{\gamma}} \widetilde{\rho}(x) ds \ge \ge 1$ for every locally rectifiable path $\widetilde{\gamma} \in f(\Gamma^*)$. Define a function $\rho : \mathbb{R}^n \to \to \mathbb{R}^n$ by setting $\rho(x) = \widetilde{\rho}(f(x)) ||f'(x)||$ if $x \in U_1 \setminus E$, $\rho(x) = \infty$ if $x \in E$, and $\rho(x) = 0$ if $x \in \mathbb{R}^n \setminus U_1$. Arguing as in the proof of the second part of Theorem 6.5 in [12] (or of Theorem 32.3 in [14], which is the n-dimensional variant of the first theorem), we further infer that $\rho \in \mathcal{R}(\Gamma^*)$, and hence

$$M_p(\Gamma) = M_p(\Gamma^*) \le \int\limits_{\mathbb{R}^n} \rho^p dx = \int\limits_{U_1} [\widetilde{\rho}(f(x))]^p ||f'(x)||^p dx =$$

$$= \int\limits_{U_1} [\widetilde{\rho}(f(x))]^p \frac{||f'(x)||^p}{|J(x,f)|} |J(x,f)| dx \le K^{p+n-2} \int\limits_{U_1} [\widetilde{\rho}(f(x))]^p |J(x,f)| dx =$$

$$= K^{p+n-2} \int\limits_{U_2} [\widetilde{\rho}(y)]^p dy = K^{p+n-2} \int\limits_{\mathbb{R}^n} [\widetilde{\rho}(y)]^p dy. \quad (2)$$

In (2), we have used the fact that, since f is K-quasi-isometry, it is easy to verify the inequality $\frac{||f'(x)||^p}{|J(x,f)|} \le K^{p+n-2}$ for $x \in U_1 \setminus E$. Taking (2) into account and recalling that the inverse mapping f^{-1} is also K-quasi-isometric, we finally get (1).

3. Sharpness of estimates (1). Suppose that $\Pi_n =]0,1[^n$, $K \in \in [1,\infty[$, and

$$f\colon x = (x_1,\dots,x_{n-1},x_n) \mapsto (Kx_1,\dots,Kx_n,K^{-1}x_n), \quad x \in \Pi_n.$$

Then $f\colon \Pi_n \to f(\Pi_n)$ is a K-quasi-isometric homeomorphism, and if $p \in]1,\infty[$ and Γ is the family of paths joining the sets $]0,1[^{n-1}\times\{0\}$ and $]0,1[^{n-1}\times\{1\}$ in Π_n, $f(\Gamma) = \{f\circ\gamma\colon \gamma\in\Gamma\}$ then $M_p(\Gamma) = 1$, and

$$M_p(f(\Gamma)) = \frac{K^{n-1}}{(K^{-1})^{p-1}} = K^{p+n-2}.$$

Thus, the rightmost estimate in (1) is sharp. Similarly, the leftmost estimate is also sharp.

Remark 3. *It is worth noting that estimates (1) were previously unknown.*

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