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ON ZEROS OF SOME ANALYTIC SPACES OF AREA NEVANLINNA TYPE IN HALFPLANE

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We introduce two new scales of spaces of Nevanlinna type in a halfplane $\mathbb{C}_+$ and provide descriptions of their zero sets and based on it we present parametric representations of these classes.

§ 1. Introduction

One of the most important directions in the theory of holomorphic functions in halfplane is the study of various problems connected with zero sets of functions of these classes (see [1], [2]). In this note we introduce new analytic classes in halfplane and provide descriptions of their zero sets in $\mathbb{C}_+$, where $\mathbb{C}_+$ is a upper halfplane in $\mathbb{C}$. We plan to provide some applications we presented in this note in a separate paper. The basic facts of the theory of analytic classes in the halfplane were proved by V. Krylov (see [5]) and E. Hille - Tamarkin independently (see also [4], [6]).

The goal of this paper is to find complete analogues of our recent results on zero sets from [7] for the case of spaces in the unit disk in a case of halfplane $\mathbb{C}_+$.

Let, as usual, $H(\mathbb{C}_+)$ is a space of all analytic functions in upper halfplane.

Let $T(f,\tau)$ (see [2]) be a Nevanlinna characteristic of f and for $0 < p < \infty$, $\alpha > -1$, let also

$$N_\alpha^1(\mathbb{C}_+) = \{f \in H(\mathbb{C}_+) : \int_0^\infty y^\alpha \int_{-\infty}^\infty \ln^+ |f(x+iy)| dxdy < \infty\}.$$

We introduce new spaces of analytic functions in $\mathbb{C}_+$.

$$(N_\alpha^p)_1 = \{f \in H(\mathbb{C}_+) :$$

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$$\int_0^\infty \left(\int_0^y \int_{-\infty}^\infty \ln^+ |f(x+i\widetilde{y})| dx d\widetilde{y} \right)^p y^\alpha dy < \infty; \}$$

$$(N^p_{\alpha,\beta})_2 = \{ f \in H(\mathbb{C}_+) :$$

$$\left(\sup_{0<y<\infty} \int_0^y \left(\int_{-\infty}^\infty \ln^+ |f(x+i\widetilde{y})| dx \right)^p \widetilde{y}^\alpha d\widetilde{y} \right) y^\beta < \infty \}, \quad \beta > 0.$$

We note that classes we introduced are Banach spaces for $p > 1,\ \alpha > -1$ and complete metric spaces for $p \le 1$.

Let further also $n(y) = n(y, \{z_k\}) = \{card\ z_k : Imz_k > y\}$, $\{z_k\}_1^\infty \in \mathbb{C}_+$. Let

$$H^p = \{F \in H(\mathbb{C}_+) : \sup_{y>0} \int_{-\infty}^\infty |F(x+iy)|^p dx \le C\},\ 0 < p < \infty.$$

The following classical result gives complete parametric representation of H^p - Hardy classes (see [4]).

Theorem 1. *Let $F \in H^p(\mathbb{C}_+)$. Then F can be represented as*

$$F(z) = e^{i\gamma} I_F(z) Q_F(z),\ z \in \mathbb{C}_+,\ \gamma \in \mathbb{R},$$

where

$$I_F(z) = \prod_{k=1}^\infty e^{i\alpha_k} \frac{z-z_k}{z-\overline{z}_k} \exp\left(\frac{1}{\pi} \int_{-\infty}^\infty \left(\frac{i}{z-t} + \frac{it}{t^2+1} \right) d\sigma(t) \right) e^{i\alpha z},$$

$F(z_k) = 0,\ e^{i\alpha_k} \dfrac{i-z_k}{i-\overline{z}_k} \ge 0,\ \sigma$ *is singular measure,* $\sigma \le 0$,

$$\int_{-\infty}^\infty \frac{|d\sigma(t)|}{1+t^2} < \infty,$$

and

$$Q_F(z) = \exp\left(\frac{1}{\pi} \int_{-\infty}^\infty \left(\frac{i}{z-t} + \frac{t}{t^2+1} \right) \log|F(t)| dt \right).$$

The goal of this note to use known Blaschke type products in a half-plane (see [2]).

$$B_\alpha(z, \{z_k\}) = \prod_{k=1}^\infty \exp\left\{ - \int_0^{2Imz_k} \frac{t^\alpha dt}{(t + i(z_k - z))^{1+\alpha}} \right\},$$

where $z, \{z_k\}_1^\infty \in \mathbb{C}_+,\ \alpha > -1,$

$$\sum_{k\geq 0}(Imz_k)^{1+\alpha} < \infty,$$

for the study of zero sets of classes we introduced above. We note also the descriptions of zero sets provide also so called parametric representation of mentioned spaces. We formulate at the end of this section a known result on zero sets for N_α^1 classes. By $Z(f)$ we denote the zero set of an analytic function f in $H(\mathbb{C}_+)$.

Theorem 2. (see [2]). *Let $\alpha > -1$. Then $N_\alpha^1(\mathbb{C}_+)$ class coincides with all functions $f,\ f \in H(\mathbb{C}_+)$ so that*

$$f(z) = CB_\alpha(z, \{a_k\}) \exp\{A_\alpha \int_{\mathbb{C}_+} \log|f(w)| \frac{|Imw|^\alpha}{(i(z-\overline{w}))^{2+\alpha}} dm_2(w)\},$$

where $A_\alpha = \dfrac{(1+\alpha)2^{1+\alpha}}{\pi}$ *and also if $Z(f) = \{a_k\}$ then*

$$\sum_{k\geq 0}(Ima_k)^{2+\alpha} < \infty. \tag{1}$$

Moreover condition (1) characterizes zero sets of $N_\alpha^1(\mathbb{C}_+)$ classes.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant, which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

§ 2. Main results

In this section we provide main results of this note and give some preliminaries that were used by us during the proof of our results.

We note that as direct corollaries of our main theorems we also formulate parametric representation of $(N_\beta^p)_1$ and $(N_{\alpha,\beta}^p)_2$ spaces in a halfplane. Various applications of such parametric representations for classes in a halfplane are well known (see [2]). Complete analogues of our parametric representations in case of classical Hardy classes in halfplane can be found in [4], [6].

Theorem 1. *Let $0 < p < \infty,\ \beta > -1$. The following is true: If $\{z_k\}_{k=1}^\infty$ is a zero set of a function $f,\ f \neq 0,\ f \in (N_\beta^p)_1$, and $\lim\limits_{t\to\infty}((t\ln|f(t)|) < \infty$.*

Then

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(\beta+2p+1)}}<\infty,\ n_k=\ n\left(\frac{1}{2^k}\right).$$

If

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(\beta+2p+1)}}<\infty,$$

then $B_\gamma(z,\{z_k\})$ *for* $\gamma>\frac{\beta+1}{p}$ *belongs to* $(N^p_\beta)_1$ *and converges uniformly within* $\mathbb{C}_+,\ 0<p<\infty,\ \beta>-1$.

Theorem 2 is an analogue of Theorem 1 for $(N^p_{\alpha,\beta})_2$ classes.

THEOREM 2. *Let* $0<p<\infty,\ \alpha>-1,\ \beta>0$. *The following are equivalent:*

(1) $\{z_k\}_{k=1}^{\infty}$ is a zero set of a function $f,\ f\neq 0, f\in(N^p_{\alpha,\beta})_2$;

(2) $n(y)\leq\dfrac{C}{y^{\alpha+\beta+p+1}}$.

Moreover, if $t>\alpha+\beta+p$ then $B_t(z,\{z_k\})$ converges uniformly within $\mathbb{C}_+$ and belongs $(N^p_{\alpha,\beta})_2$.

In following Theorem 3 and Theorem 4 we provide complete parametric representations of $\left(N^p_\beta\right)_1$ and $\left(N^p_{\alpha,\beta}\right)_2$ classes based on descriptions of zero sets we provided above for these classes in Theorem 1 and Theorem 2. We have as immediate corollaries of Theorem 1 and Theorem 2.

THEOREM 3. *Let* $0<p<\infty,\ \beta>-1,\ \lim\limits_{t\to+\infty}((t\ln|f(it)|)<\infty$, *and* $f\in(N^p_\beta)_1$, f *is a non zero function. Then* $(N^p_\beta)_1$ *coincides with a class of the functions that admits representation* $f(z)=B_\gamma(z,\{z_k\})(\exp h(z)),\ z\in\mathbb{C}_+$ *where* $\{z_k\}$ *is an arbitrary sequence from* $\mathbb{C}_+$ *that satisfies*

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(\beta+2p+1)}}<\infty,$$

$\gamma>\dfrac{\beta+1}{p}$ *and* $h\in H(\mathbb{C}_+)$, *moreover*

$$\int_0^{\infty}\left(\int_0^{y}\int_{-\infty}^{\infty}|h(x+i\widetilde{y})|dxd\widetilde{y}\right)^p y^\beta dy<\infty,\ \beta>-1.$$

THEOREM 4. *Let* $0<p<\infty,\ \alpha>-1,\ \beta>0,\ \gamma>\alpha+\beta+p,\ f\neq 0, f\in(N^p_{\alpha,\beta})_2$. *Then* $(N^p_{\alpha,\beta})_2$ *coincides with a class of functions that* *admits representation* $f(z) = B_\gamma(z, \{z_k\})(\exp h(z)),\ z \in \mathbb{C}_+$, *where* $\{z_k\}$ *is an arbitrary sequence from* $\mathbb{C}_+$ *that satisfies* $n(y) < \dfrac{C}{y^{\alpha+\beta+p+1}}$ *and* $h \in H(\mathbb{C}_+)$, *moreover*

$$\sup_{0<y<\infty} y^\beta \int_0^y \left(\int_{-\infty}^{\infty} |h(x+i\widetilde{y})|dx \right)^p \widetilde{y}^\alpha d\widetilde{y} < \infty.$$

REMARK. *Let* $w > 0$ *on* $(0, \infty)$ *and* $w \in L^1(0, A)$ *for every* $A > 0$. *Let assume for every* $A > 0$ *there are numbers* $m_\omega,\ M_\omega,\ q_\omega$, *so that* $m_\omega, q_\omega \in (0,1),\ M_\omega > 0$ *and so that* $m_\omega \le \dfrac{\omega(\lambda y)}{\omega(y)} \le M_\omega,\ \lambda \in\ [q_\omega, 1],\ y > 0$. *We finally note that results can be extended to* $w(y)$ *more general weights so that* y^α *replaced by general function* $w(y)$. *We among other things for the proofs use the following lemma which is interesting by itself.*

LEMMA 1. *Let* $\{z_n\}_1^\infty$ *be a sequence from* $\mathbb{C}_+$ *for which*

$$\sum_{n=1}^{\infty} (Imz_n)^{1+\beta} < \infty$$

and for some constant $C > 0$ *and* $\alpha > 1$ $n(y, \{z_k\}) \le Cy^{-\alpha},\ y > 0$. *Then for* $p \ge 1,\ \beta > \alpha - 1$,

$$\int_{-\infty}^{\infty} |\ln B_\beta(x+iy, \{z_k\})|^p dx \le K(\alpha, \beta, p)C^p y^{1-\alpha p},$$

for some constant K.

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