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ON A LENGTH OF THE CONTINUED FRACTION'S PERIOD

В статье дана оценка количества чисел d из отрезка натурального ряда, для которых непрерывная дробь для $\sqrt{d}$ имеет большой период.

Let $L(d)$ be the length of the continued fraction's period of $\sqrt{d}$ for d that isn't a square of an integer. Let p be a prime number, N, P, Q, k, r, s – natural numbers.

From the results of E.V.Podsypanin [1] it follows

$$L(d) = O(\sqrt{d}\log d).$$

From the results of E.P.Golubeva [2] it follows

$$L(d) = O(\sqrt{d}\,L(1,\chi)\,2^{-\omega(d)}),$$

where χ is a character of the quadratic field $\mathbb{Q}(\sqrt{d})$, $\omega(d)$ is a number of prime divisors of d.

In the paper [3] has been proved that for any real $K > 0$ and for sufficiently great N the number of integers d, $N < d \le 2N$, for which $L(d) \ge K\sqrt{d}$ don't exceed $cN/\log K$ with absolute constant c.

The irrational

$$\xi = \frac{-P + \sqrt{d}}{Q} \qquad (1)$$

where P and Q sutisfy to the congruence

$$P^2 \equiv d \pmod{Q} \qquad (2)$$

is called reduced if $0 < \xi < 1$ and conjugate number $\xi' < -1$, $\xi' = (P - \sqrt{d})/Q$. By other words, ξ is reduced if

$$0 < P < \sqrt{d},\ \sqrt{d} - P < Q < \sqrt{d} + P \qquad (3)$$

Let's note that the continued fraction's remainders have form (1) and satisfy to conditions (2) and (3).

Let $H(d)$ denote a number of the all reduced irrationals of form (1) with the same d. The quantity $H(d)$ equel to the lengthes sum of the not equivalent in pairs quadratic irrational's periods.

In this paper will be proved the following theorems.

Theorem 1. *For any N*

$$\frac{6}{\pi^2}N^{3/2} + O(N) \le \sum_{d=N+1}^{2N} H(d) \le 4\sqrt{2}N^{3/2}$$

with the absolute constant into the symbol O.

Theorem 2. *For any N and real $K > 0$*

$$\#\{d \mid N < d \le 2N, L(d) \ge K\sqrt{d}\} < \frac{4\sqrt{2}}{K}N.$$

Proof of Theorem 1. The quantity $H(d)$ equal to a number of all pairs (P,Q) of P and Q satisfying to inequalities (3) and to congruence (2).

Let $\rho(d,Q)$ denote the number of solutions of congruence (2). Then

$$\sum_{d=N+1}^{2N} H(d) \le \sum_{d=N+1}^{2N} \sum_{0<Q<2\sqrt{d}} \rho(d,Q).$$

Let's change the order of summing in the right part:

$$\sum_{d=N+1}^{2N} H(d) \le \sum_{0<Q<2\sqrt{2N}} \sum_{d=N+1}^{2N} \rho(d,Q) \tag{4}$$

We have increased the region of the variation of d for values $Q \ge \sqrt{2N}$. This permutation of the sums as a matter of fact is using of the quadratic reciprocity law.

Let's consider the interior sum of the right part of (4).

$$\sum_{d=N+1}^{2N} \rho(d,Q) \leq \sum_{k=1}^{[N/Q]+1} \sum_{d=N+(k-1)Q+1}^{N+kQ} \rho(d,Q) \qquad (5)$$

where $[x]$ is an entire of real x. Let $S(k,Q)$ be the interior sum of the right part of (5). Let $\delta = (d,Q) = rs^2$, $d_1 = d/\delta, Q_1 = Q/\delta$, where r is a square-free integer. We are need in the following lemma now.

Lemma. *Congruence (2) is solvable if and only if d is a quadratic residue $modQ_1$ and $(r,Q_1) = 1$. In this case there exist s solutions of congruence (2)for every solution of the congruence*

$$rx^2 \equiv d_1 \,(modQ_1) \qquad (6)$$

Proof. An existance of a solution of congruence (6) is nesessary and sufficient for a solvable of congruence (2). Let (6) is solvable. Then $(r,Q_1) = 1$. If r' is an inverse element for r in the multiplicative group in the ring of residue classes $modQ_1$ then d_1r' is a quadratic residue $modQ_1$. But r' is a quadratic residue if and onli if r is a quadratic residue. Hence d_1r is a quadratic residue. Thus d is a quadratic residue $modQ_1$.

Inversly, if d is a quadratic residue $modQ_1$ then d/s^2 is the same thing. Let's suppose that $(r,Q_1) = 1$. Then there exists r' that $rr' \equiv 1(modQ_1)$ and d_1r' is a quadratic residue $modQ_1$. Therefore, congruence (6) is solvables and congruence (2) is solvables too. The first statement of the lemma has proved.

Let x_0 be any solution of congruence (6). Put

$$P = rsx_0 + krsQ_1, \; 0 \leq k < s. \qquad (7)$$

It's clear that these numbers are not congruenced $modQ$. Furthermore

$$P^2 = (rsx_0)^2 + 2rx_0Qk + k^2rx_0QQ_1 \equiv (rsx_0)^2 \equiv d\,(modQ).$$

Thus for every solution x_0 of congruence (6) formula (7) gives the s distinguish solutions of congruence (2). The lemma has proved.

Using the lemma we'll find the quantity $n(\delta, Q)$ of the numbers d with the fixed $\delta = (d, Q)$ for which congruence (2) is solvables.

Owing to the square-free factor r of the integer d is fixed the number such d is defined by the number of quadratic residues or non-residues $modQ_1$ provided that r is a quadratic residue or non-residue correspondently.

For every prime factor p of Q_1 there are $\frac{p-1}{2}$ such numbers $modp$ and there are $\frac{\phi(p^\alpha)}{2}$ such numbers $modp^\alpha$. There are $\phi(Q_1)/2^{\omega(Q_1)}$ such numbers $modQ_1$ for an odd integer Q_1. Here $\phi(n)$ is the Euler's fonction and $\omega(n)$ is a number of prime factors of an integer n.

If Q_1 containes 2^k with maximum k, then the correspondent conditions have the form:

$$d \equiv r(mod2^m),\ m = min\{k, 3\}.$$

Hence

$$n(\delta, Q) = \phi(Q_1)2^{-\omega(Q_1)}\ if\ Q_1\ is\ odd\ or\ Q_1 \equiv 4\,(mod8),$$

$$n(\delta, Q) = \phi(Q_1)2^{-\omega(Q_1)+1}\ if\ Q_1 \equiv 2\,(mod4),$$

$$n(\delta, Q) = \phi(Q_1)2^{-\omega(Q_1)-1}\ if\ Q_1 \equiv 0\,(mod8).$$

If for given d and Q congruence (2) is solvable then owing to the lemma for a number of the solutions we shell have:

$$\rho(d, Q) = 2^{\omega(Q_1)}s\ \ if\ Q_1 \equiv 1\,(mod2)\ or\ Q_1 \equiv 4\,(mod8),$$

$$\rho(d, Q) = 2^{\omega(Q_1)-1}s\ \ if\ Q_1 \equiv 2\,(mod4),$$

$$\rho(d, Q) = 2^{\omega(Q_1)+1}s\ \ if\ Q_1 \equiv 0\,(mod8).$$

From this we get:

$$S(k, Q) = \sum_{rs^2|Q,(r,Q/rs^2)=1} s\phi(\frac{Q}{rs^2}). \tag{8}$$

Let's denote u the production of the primes p which divide Q but p^2 don't divide Q and $v = Q/u$. Then

$$S(k,Q) = \sum_{s^2|v}\sum_{r|u}\phi(\frac{u}{r}) \sum_{t|\frac{v}{s^2}}^{(t,v/ts^2)=1} s\phi(\frac{v}{ts^2}) =$$

$$u\sum_{s^2|v}\sum_{t|v/s^2}^{(t,v/s^2)=1} s\phi(\frac{v}{ts^2}) \le u\sum_{d|v}\phi(\frac{v}{d}) = uv = Q.$$

Owing to (4) and (5) we find:

$$\sum_{d=N+1}^{2N} H(d) \le N \sum_{Q\le 2\sqrt{2N}} (1+\frac{Q}{N}) \le 4\sqrt{2}N^{3/2}.$$

The right part of the inequality of theorem 1 has proved.

For the the proof of the left part inequality we shell take into account only that pairs (P,Q) in the reduction region that satisfy to the conditions

$$0 < Q < \sqrt{d},\ \sqrt{d} - Q < P < \sqrt{d}.$$

In this case

$$\sum_{d=N+1}^{2N} H(d) \ge \sum_{0<Q<\sqrt{2N}} \sum_{d=N+1,(d,Q)=1}^{2N} \rho(d,Q).$$

The interior sum is estimated similarly to the same thing in (4):

$$\sum_{N<d<2N}^{(d,N)} \rho(d,Q) \ge [N/Q]\phi(Q).$$

Therefore

$$\sum_{d=N+1}^{2N} H(d) \ge \sum_{0<Q<\sqrt{2N}} [N/Q]\phi(Q) = \frac{6}{\pi^2}N^{3/2} + O(N).$$

The theorem has proved.

Proof of theorem 2. Owing to $L(d) \leq H(d)$ theorem 1 implies

$$\sum_{d=N+1}^{2N} L(d) \leq 4\sqrt{2}N^{3/2}. \tag{9}$$

Let's choose any real K and denote n for a number of d in (9) for that $L(d) \geq K\sqrt{d}$. Then inequality (9) implies

$$nK\sqrt{N} \leq 4\sqrt{2}N^{3/2}.$$

From this inequality it follows the statement of theorem 2.

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