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RIEMANN BOUNDARY-VALUE PROBLEM ON CASSINI SPIRALS

Abstract. Recently, a number of papers has been published concerning the Riemann boundary-value problems on spiral-like arcs. In all these works, the spiral has the shape close to concentric circles, i. e., with equal rates of torsion in all directions. This paper is the first study of this problem on spirals with oval turns.

Key words: *Rimann boundary-value problem, non-rectifiable curve, spirals, strong torsion, Cassini ovals.*

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1. Introduction. We consider the Riemann boundary-value problem on an arc Γ . Its statement is well known. Let Γ be a directed simple Jordan arc beginning at a point a_1 and ending at a point a_2 . We seek for all holomorphic in $\bar{\mathbb{C}} \setminus \Gamma$ functions $\Phi(z)$, such that

(RP1) they have limit values $\Phi^\pm(t)$ from the left and from the right at any point $t \in \Gamma' := \Gamma \setminus \{a_{1,2}\}$;

(RP2) these boundary values are connected by relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),$$

where $G(t)$ and $g(t)$ are given functions;

(RP3) $\Phi(\infty) = 0$;

(RP4) the desired functions $\Phi(z)$ satisfy the following bounds near the end-points $a_{1,2}$:

$$\Phi(z) = O(|z - a_j|^{-\gamma_j}), \quad 0 \leq \gamma_j < 1, \quad j = 1, 2.$$

There exists a great body of papers on this problem. In the classical works (see, for instance, [1–3]) the arc Γ is assumed piecewise-smooth.

Later, the Riemann problem was solved on non-rectifiable arcs [4–6]. However, these papers contain additional restrictions on behavior of the arc Γ at its end-points. In order to describe these restrictions, we introduce the so called logarithmic kernel of the arc Γ :

$$K_{\Gamma}(z) := \frac{1}{2\pi i} \ln \frac{z - a_2}{z - a_1}.$$

It is a single-valued branch of logarithm, holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$, and uniquely determined by condition $K_{\Gamma}(\infty) = 0$. Clearly,

$$K_{\Gamma}(z) = A_{\Gamma}(z) + \frac{1}{2\pi i} \ln \left| \frac{z - a_2}{z - a_1} \right|,$$

where $A_{\Gamma}(z)$ is a real-valued function. Generally speaking, singularities of arbitrarily high orders at the points $a_{1,2}$ are allowed. The Riemann boundary-value problem on non-rectifiable arcs is solved in [4], [6] under one of the following assumptions:

- if $G(t) \equiv 1$ (the so called jump problem) and $K_{\Gamma}(z)$ is square integrable near the points $a_{1,2}$;
- if $g(t) \equiv 0$ (the homogeneous problem) and $K_{\Gamma}(z) = O(-\ln |z - a_j|)$ near the points a_j , $j = 1, 2$;
- if the limits

$$\lim_{z \rightarrow a_j} \frac{K_{\Gamma}(z)}{\ln |z - a_j|}, \quad j = 1, 2.$$

exist.

Recently, two articles [7], [8] have been published; they deal with the Riemann problems on arcs with singularities of logarithmic kernels of higher orders. The arc is called spiral of strong torsion if its logarithmic kernel is not integrable near one of its end-points. The solvability of the Riemann boundary-value problem on such arcs essentially depends on their shapes at the ends. In this paper, we study the Riemann boundary-value problem on spirals of strong torsion with Cassini oval-like turns.

2. Cassini spirals. Cassini ovals¹ are domains determined on the complex plane by the relation

$$C(p, q) = \{z : |z^2 - q^2| < p^2\}, \quad p > q > 0.$$

¹named after the famous astronomer and mathematician Giovanni Domenico Cassini, 1625-1712

In other words, it is the set of points such that the product of their distances from the focal points $\pm q$ is less than p^2 .

Note that equation of the boundary of such oval can be written as

$$\bar{z}^2 = q^2 + \frac{p^4}{z^2 - q^2}. \quad (1)$$

Let $C(p_{1,2}, q_{1,2})$ be two Cassini ovals. Elementary transformations show that if $q_1 < q_2$ and $q_2^2 - q_1^2 < p_2^2 - p_1^2$, then $C(p_1, q_1) \subset C(p_2, q_2)$.

Fix two decreasing sequences of positive values $\mathbf{p} = \{p_1, p_2, \dots\}$ and $\mathbf{q} = \{q_1, q_2, \dots\}$, such that $q_{n+1}^2 - q_n^2 < p_{n+1}^2 - p_n^2$ for any positive integer n and $\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n = 0$, and consider the sequence of consecutively embedded Cassini ovals $C_n = C(p_n, q_n)$, $n = 1, 2, \dots$, such that $\bigcap_{n=1}^{\infty} C_n = \{0\}$. Denote $R_n := C_n \setminus C_{n+1}$. Let x_n be the point of intersection between boundary of the oval C_n with the positive ray of the real axis $R_+ = [0, +\infty)$.

In each domain R_n , we consider Jordan arcs γ_n beginning at the point x_{n+1} and ending at x_n . The domain Δ_n to the left of the arc (respectively, to the right) has the boundary $\gamma_n \cup [x_{n+1}, x_n]$. Let the arcs be such that this domain contains the oval C_{n+1} . The union of these arcs and the point 0

$$\Gamma = \left(\bigcup_{n=0}^{\infty} \gamma_n \right) \cup \{0\}$$

is called the left (respectively, right) Cassini spiral. It is directed from the point 0 to the point x_1 . We define the signature of the spiral as the value σ_Γ that equals +1 or -1 for the left and right spiral, correspondingly.

If Γ is a Cassini spiral, then the argument of its logarithmic kernel has the asymptotic estimations

$$A_\Gamma(z) = \sigma_\Gamma \sum_{n=1}^{+\infty} \chi_n(z) + O(1), \quad z \rightarrow 0,$$

$$A_\Gamma(z) = O(1), \quad z \rightarrow x_1,$$

where χ_n is the characteristic function of the oval C_n .

The oval $C(p, q)$ satisfies the inclusion

$$\{z : |z| < \sqrt{p^2 - q^2}\} \subset C(p, q) \subset \{z : |z| < \sqrt{p^2 + q^2}\}.$$

Hence, if

$$\sum_{n=1}^{+\infty} (p_n^2 - q_n^2) = \infty, \quad (2)$$

then the Cassini spiral has strong torsion at the origin. For instance, this concerns the spiral with $q_n = n^{-\alpha}$, $p_n = kq_n$, $0 < \alpha \leq 1/2$, $k > 1$.

If arc γ_n is rectifiable, then its length $|\gamma_n|$ exceeds $2\pi\sqrt{p_n^2 - q_n^2}$, and, under condition (2), the total length of Γ is infinite.

In what follows, we assume, for the sake of brevity, that Γ satisfies the so called Luzin condition, i. e., any portion of Γ contains a rectifiable subarc. As noted by N. I. Luzin, a curve with this property is erasable in the class of continuous functions (see, for instance, [9]). Clearly, a curve satisfying Luzin condition has null plane measure.

3. The jump problem. We consider first the jump problem, i. e., the problem (RP1)–(RP4) with $G \equiv 1$. In the piecewise smooth case, its solution is given by the Cauchy type integral, but Cassini spirals of strong torsion cannot be rectifiable, and this integral diverges. In order to build a solution, we use the following functional classes.

As customary, the Lipschitz space $\text{Lip } \Gamma$ consists of functions f defined on Γ and satisfying the condition

$$\sup \left\{ \frac{|f(t) - f(t')|}{|t - t'|} : t, t' \in \Gamma, t \neq t' \right\} := \Lambda(f) < +\infty.$$

The Taylor space $T^m(\Gamma)$ consists of functions f representable as

$$f(\tau) = P_f(\tau, \bar{\tau}) + \tau^m f_0(\tau), \quad f_0 \in \text{Lip } \Gamma,$$

where $P_f(\tau, \bar{\tau})$ is the Taylor polynomial of degree less than m .

We use its subspaces: $T_0^m(\Gamma)$ consists of functions $f \in T^m(\Gamma)$ with null Taylor polynomials; $T_H^m(\Gamma)$ consists of functions $f \in T^m(\Gamma)$ with holomorphic Taylor polynomials; $T_E^m(\Gamma)$ consists of functions $f \in T^m(\Gamma)$, such that the Taylor polynomial $P_f(\tau, \bar{\tau})$ contains only even degrees of $\bar{\tau}$.

Let a Cassini spiral Γ satisfy the bound

$$K_\Gamma(z) = O(|z|^{-m}), \quad z \rightarrow 0, \tag{3}$$

and $g \in T_0^m(\Gamma)$. Consider the function

$$U(z) := K_\Gamma(z)z^m \mathcal{E}g_0(z)\psi(z),$$

where \mathcal{E} is the Whitney extension operator (see, for instance, [10]), and $\psi(z)$ is a smooth function with compact support; it equals unit in a neighborhood of Γ . The function

$$\Phi_0(z) := U(z) - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial U}{\partial \bar{\zeta}} \frac{d\zeta d\bar{\zeta}}{\zeta - z}$$

is holomorphic in $\mathbb{C} \setminus \Gamma$, vanishes at the infinity, has a jump g on Γ , and satisfies the bounds

$$\Phi_0(z) = O(1), z \rightarrow 0, \quad \Phi_0(z) = g(x_1)K_\gamma(z), z \rightarrow x_1 \tag{4}$$

at the end-points of the arc. Hence, it is a solution of the jump in this case.

Then, consider the case $g \in T_H^m(\Gamma)$ and the function

$$\Phi_1(z) := \Phi_0(z) + P_g(z)K_\Gamma(z) - Q(z),$$

where $Q(z)$ is a holomorphic polynomial, such that the function Φ_1 vanishes at the infinity. This polynomial exists and is unique. The function Φ_1 satisfies conditions (RP1)–(RP3), but the bounds (4) here turn into the asymptotic formulae

$$\Phi_1(z) = P_g(z)K_\Gamma(z) + O(1), z \rightarrow 0, \quad \Phi_1(z) = g(x_1)K_\gamma(z), z \rightarrow x_1.$$

Hence, the problem with this jump becomes resolvable if we replace the condition (RP4) by the bounds

$$\Phi(z) = O(z^{-m}), z \rightarrow 0, \quad \Phi(z) = O(|z - x_1|^{-\gamma}), z \rightarrow x_1, 0 \leq \gamma < 1,$$

which we call condition (RP5). The general solution can be written as

$$\Phi(z) = \Phi_1(z) + z^{-m}P(z), \tag{5}$$

where $P(z)$ is an arbitrary polynomial of degree less than m .

Let $g \in T_E^m(\Gamma)$, i. e.,

$$g(t) = t^m g_0(t) + P_g(t, \bar{t}), \quad P_g(t, \bar{t}) = \sum_{\alpha, \beta \geq 0}^{\alpha + 2\beta < m} c_{\alpha, 2\beta} t^\alpha \bar{t}^{2\beta},$$

where $g_0 \in \text{Lip } \Gamma$, α and β are non-negative integers, and $c_{\alpha, \beta}$ are constants. The function Φ_0 has a jump $t^m g_0(t)$. It remains to construct a function with the jump $P_g(t, \bar{t})$. Connect the end-points of the spiral Γ with the segment $I := [0, a_1]$ and consider three jump problems:

$$\Phi_{21}^+(\tau) - \Phi_{21}^-(\tau) = \sigma P(\tau, \bar{\tau}), \quad \tau \in \cup_{n=1}^\infty \partial \Delta'_n,$$

$$\Phi_{22}^+(\tau) - \Phi_{22}^-(\tau) = P_g(\tau, \bar{\tau}), \quad \tau \in I,$$

$$\Phi_3^+(\tau) - \Phi_3^-(\tau) = \sigma P_g(\tau, \bar{\tau}), \quad \tau \in \cup_{n=1}^\infty C_n.$$

The contour of the first problem is the union of boundaries of the domains $\Delta'_n := \Delta_n \setminus C_{n+1}$. As these domains do not overlap, one of the solutions is the generalized Cauchy-type integral

$$\Phi_1(z) := \sigma P(z, \bar{z}) - \frac{\sigma}{2\pi i} \iint_{\cup_{n=1}^\infty \partial \Delta'_n} \frac{\partial P}{\partial \bar{w}} \frac{dw d\bar{w}}{w - z},$$

which is bounded in the whole complex plane and vanishes at the infinity.

The solution of the second problem is the customary integral of the Cauchy type:

$$\Phi_2(z) = \frac{1}{2\pi i} \int_I \frac{P(\tau, \bar{\tau}) d\tau}{\tau - z},$$

with well-known properties.

Finally, the contour of the third problem is the infinite set of boundaries of the Cassini ovals C_n . By virtue of the formula (1), the sum

$$S(z) := \sum_{n=1}^\infty \chi_n(z) \sum_{\substack{\alpha+2\beta < m \\ \alpha, \beta \geq 0}} c_{\alpha, 2\beta} z^\alpha \left(q_n^2 + \frac{p_n^4}{z^2 - q_n^2} \right)^\beta$$

satisfies the boundary-value condition of the jump problem. It has poles at points $\pm q_n$, but we can "erase" them by using the Mittag-Leffler theorem (see, for instance, [11]). According to this theorem, there exists a function $M(z)$, meromorphic in $\bar{\mathbb{C}} \setminus \{0\}$, such that it has poles at the points $\pm q_n$ only, and its main parts at the poles coincide with the main parts of $S(z)$. Then the sum

$$\Phi_*(z) = \Phi_0(z) + \Phi_1(z) + \Phi_2(z) + S(z) - M(z)$$

satisfies the conditions (RP1)–(RP3). Its asymptotic at the origin is unknown.

Thus, we have proved the following

Theorem 1. *Let the Cassini spiral of strong torsion Γ satisfy Luzin's condition and the bound (3). Then the following results are valid:*

If $g \in T_0^m(\Gamma)$, then the jump problem (RP1)–(RP4) has a unique solution.

If $g \in T_H^m(\Gamma)$, then the jump problem (RP1), (RP1), (RP3), (RP5) is resolvable, and its general solution is (5), where $P(z)$ is an arbitrary polynomial of degree less than m .

If $g \in T_E^m(\Gamma)$, then the jump problem (RP1)–(RP3) is resolvable, and its general solution is $\Phi_*(z) + F(z^{-1})$, where $F(w)$ is an arbitrary entire function, such that $F(0) = 0$.

4. Homogeneous problem. We consider here the homogeneous Riemann boundary-value problem on a Cassini spiral of strong torsion, putting $g \equiv 0$, $G(t) = \exp f(t)$, $f \in T^m(\Gamma)$.

Assume first that $f \in T_0^m(\Gamma)$. Let $\phi_0(z)$ be a solution of the jump problem with the jump $f(t)$, i. e.,

$$\phi(z) = u(z) - \frac{1}{2\pi i} \iint_{\mathcal{C}} \frac{\partial u}{\partial \bar{\zeta}} \frac{d\zeta d\bar{\zeta}}{\zeta - z},$$

where $u(z) := K_\Gamma(z)z^m \mathcal{E}f_0(z)\psi(z)$. Denote $X_0(z) = \exp \phi(z)$. Obviously,

$$X_0^+(t) = G(t)X_0^-(t), \quad t \in \Gamma'. \quad (6)$$

This function is bounded near the origin, and near end-point of Γ we have

$$\ln |X_0(z)| = \operatorname{Re} f(x_1)K_\Gamma(z) + O(1).$$

We put $f(x_1) = a + ib$, where a and b are real. Then

$$\ln |X_0(z)| = (2\pi)^{-1}b \ln |z - x_1| + O(1), \quad z \rightarrow x_1,$$

because the argument A_Γ is bounded near x_1 . Denote by \varkappa the entire part of $(2\pi)^{-1}b$, and put $X(z) = (z - x_1)^{-\varkappa}X_0(z)$. Clearly, for $\varkappa \leq 0$ this homogeneous Riemann problem has the null solution only, and for $\varkappa > 0$ its general solution is $\Phi(z) = X(z)P(z)$, where P is an arbitrary polynomial of degree less than \varkappa .

Now we consider the case $f \in T_H^m(\Gamma)$. The corresponding solution of the jump problem is

$$\phi_1(z) := \phi_0(z) + P_f(z)K_\Gamma(z) - Q(z),$$

where $Q(z)$ is the polynomial, such that ϕ_1 vanishes at the infinity point. We put $X_0(z) = \exp \phi_1(z)$. Clearly, it satisfies the boundary-value condition (6) and the estimate (). Therefore, the function

$X(z) := (z - x_1)^{-\varkappa} X_0(z)$ has the asymptotic $X(z) \asymp (z - x_1)^{-\gamma}$, $0 \leq \gamma < 1$, at the point x_1 . Near the origin, we have

$$\ln |X(z)| = A_\Gamma(z) \operatorname{Re} P_f(z) + (2\pi)^{-1} \ln |z| \operatorname{Im} P_f(z) + O(1).$$

Let $\Phi(z)$ be a solution of the homogeneous problem (RP1)–(RP4). Then the ratio $F(z) = \Phi(z)/X_0(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \{0, \infty\}$, has order $\varkappa - 1$ at ∞ , and satisfies the bound

$$|F(z)| \asymp \exp(-A_\Gamma(z) \operatorname{Re} P_f(z))$$

near the origin. Here we restrict ourselves by the case where the polynomial $P_f(z)$ has null of positive order $k < m$ at the origin, i. e.,

$$P_f(z) = \sum_{n=k}^{m-1} c_n z^n, \quad c_k \neq 0.$$

Then $A_\Gamma(z) \operatorname{Re} P_f(z) = O(|z|^{k-m})$. On the other hand, the function $F(z)$ is bounded on the set $Z := \{z : \operatorname{Re} P_f(z) = 0\}$. This set contains $2k$ arcs beginning at the origin, and these arcs divide a neighborhood of the origin on curvilinear sectors with angles π/k at the origin. If $k > m - k$, then F is holomorphic at the origin due to the Phragmen-Lindelöf principle (see, for instance, [11]), and, consequently, $F \equiv 0$.

The results of the previous section enable us to solve also the homogeneous Riemann problem on a Cassini spiral for $f \in T_E^m(\Gamma)$, but we do not have a bound for its solution at the origin. In other words, the obtained solution satisfies the conditions (R1)–(R3) only. Thus, the following theorem is proved:

Theorem 2. *Let Cassini spiral of strong torsion Γ satisfy Luzin’s condition and the bound (3). Then the following results concerning the homogeneous Riemann problem are valid:*

If $G = \exp f$, $f \in T_0^m(\Gamma)$, then the homogeneous problem (RP1)–(RP4) for $\varkappa > 0$ has a general solution $\Phi(z) = X(z)P(z)$, where P is an arbitrary polynomial of degree less than \varkappa , and it has the null solution only for $\varkappa \leq 0$.

If $G = \exp f$, $f \in T_H^m(\Gamma)$ and the polynomial $P_f(z)$ has a root of order k at the origin, $m > k > m/2$, then the homogeneous problem (RP1)–(RP4) has the null solution only.

If $G = \exp f$, $f \in T_E^m(\Gamma)$, then the homogeneous problem (RP1)–(RP3) is resolvable.

5. Inhomogeneous problem. Consider now one case of the full Riemann boundary-value problem on a spiral. Let $G = \exp f$, $f \in T_0^m(\Gamma)$, and $g \in T_0^m(\Gamma)$. Then the customary factorization technique enables us to imply the following result:

Theorem 3. *Let the Cassini spiral of strong torsion Γ satisfy Luzin's condition and the bound (3), $G = \exp f$, $f \in T_0^m(\Gamma)$, and $g \in T_0^m(\Gamma)$. Then, for $\varkappa > 0$, the inhomogeneous problem (RP1)–(RP4) has the general solution $\Phi(z) = \Psi(z) + X(z)P(z)$, where $\Psi(z)$ is its special solution and P is an arbitrary polynomial of degree less than \varkappa . For $\varkappa \leq 0$, it has the unique solution under $-\varkappa$ solvability conditions.*

We see that in this case the inhomogeneous Riemann boundary-value problem on Cassini spirals has the same solvability pattern as the problem on piecewise-smooth arcs [1–3]. Probably, its solution for boundary data in spaces $T_E^m(\Gamma)$ and $T_H^m(\Gamma)$ is more sophisticated.

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