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PERTURBED COMPANIONS OF OSTROWSKI TYPE INEQUALITIES FOR N-TIMES DIFFERENTIABLE FUNCTIONS AND APPLICATIONS

Abstract. We firstly examine some inequalities obtained by using sets of complex-valued functions for functions whose high order derivatives are restricted. We also give some approximations for the functions whose derivatives up to the order $n-1$ $(n \geq 1)$ are continuous and whose the nth derivatives are of bounded variation. So, the results provide extensions of those presented in earlier works.

Key words: *Function of bounded variation, Perturbed Ostrowski type inequalities*

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1. Introduction. The inequality discovered by Ostrowski in 1938 has been studied by a large number of researchers due to its comprehensive application fields in numerical analysis and certain special means. This inequality [21], established by using mappings whose first derivatives are bounded, is stated in the following manner.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i. e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty , \qquad (1)$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Over the years, interested researchers have studied it to provide novel refinements, improvements, and generalizations of the inequality (1). For instance, some authors deduced new Ostrowski-type inequalities for differentiable, twice differentiable, or higher-order differentiable functions in [7], [8], [9], and [22] (see also references therein). On the other side, the perturbed method has been much used to generalise integral inequalities. For example, after Dragomir had published his paper [14] involving the perturbed inequality of the Ostrowski type established by utilizing absolutely continuous functions, some authors focused on perturbed integral inequalities for twice and higher order differentiable mappings in [5], [16], [17], [18], and [19]. What is more, some companion perturbed inequalities for various assumptions of the functions are refined by using three- and five-step quadratic kernels in [15], [23], and [24].

In particular, some mathematicians focus on the Ostrowski-type inequalities obtained by using mappings of bounded variation, as well as the other function species. In the reference [11], Dragomir introduced the following useful result for functions of bounded variation:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

Morever, Dragomir indicated the original generalisation of the Ostrowski-type results for functions that are of bounded variation in [10]. Afterwards, results pertaining to the inequality (1) for functions whose first derivatives are of bounded variation, are given in [1], [6], and [20]. Also, certain generalized outcomes for mappings that possess n-th derivatives of bounded variation, are established in [3] and [13]. In addition to all the results, some companion versions of perturbed results concerning Ostrowski's inequality for bounded-variation mappings are examined in [2], [4], and [12].

We also note that Dragomir established the following identity, so as to observe some perturbed outcomes of Ostrowski-type inequalities in [14].

Theorem 3. *Let $f : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$ and $x \in [a,b]$. Then, for any complex numbers $\lambda_1(x)$ and $\lambda_2(x)$, we have*

$$\frac{1}{b-a} \int_a^x (t-a) \left[f'(t) - \lambda_1(x) \right] dt + \frac{1}{b-a} \int_x^b (t-b) \left[f'(t) - \lambda_2(x) \right] dt =$$

$$= f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f'(t)dt$$

where the integrals in the left-hand side are taken in the Lebesgue sense.

The primary purpose of this work is to deduce original inequalities for functions, whose higher-order derivatives are limited. For this, some approximations are examined with the help of the identity obtained by utilizing higher-order differentiable mappings. So, the new companion results are derived, regarding inequality (1) for functions whose n-th derivatives are bounded and of bounded variation. Relations between these results and inequalities given in the earlier works are also examined.

2. The case when $f^{(n)}$ is bounded. Before we can establish the inequalities that will be given in this section, we should mention the following identity.

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an n-time differentiable function on (a,b). Then, for any complex numbers $\lambda_i(x)$, $i = 1,2,3$ and all $x \in [a, \frac{a+b}{2}]$, we have the identity*

$$\int_a^x \frac{(t-a)^n}{n!}\left[f^{(n)}(t) - \lambda_1(x)\right] dt +$$
$$+ \int_x^{a+b-x} \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n \left[f^{(n)}(t) - \lambda_2(x)\right] dt +$$
$$+ \int_{a+b-x}^b \frac{(t-b)^n}{n!}\left[f^{(n)}(t) - \lambda_3(x)\right] dt =$$
$$= S(f:n,x) - R(n,x) + (-1)^n \int_a^b f(t)dt, \quad (3)$$

where $S(f:n,x)$ and $R(n,x)$ are defined by

$$S(f:n,x) = \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[f^{(k)}(a+b-x) + (-1)^k f^{(k)}(x)\right]}{(k+1)!} \times$$
$$\times \left[(x-a)^{k+1} + (-1)^k \left(\frac{a+b}{2} - x\right)^{k+1}\right] \quad (4)$$

and

$$R(n,x) = [\lambda_1(x) + (-1)^n \lambda_3(x)] \tfrac{(x-a)^{n+1}}{(n+1)!} + [1+(-1)^n] \tfrac{\lambda_2(x)}{(n+1)!} (\tfrac{a+b}{2} - x)^{n+1}.$$

Proof. Combining the resulting identities by using fundamental analysis operators, after applying integration by parts n times to the three integrals in the right-hand side of the equality (3), the required identity can be easily obtained. □

The expression $S(f:n,x)$ (4) will be used throughout this paper.

Furthermore, we define the sets of complex-valued mappings, for $\gamma, \Gamma \in \mathbb{C}$ and an interval of real numbers $[a,b]$,

$$\overline{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \,\middle|\, \Re\left[(\Gamma - f(t)) \left(\overline{f(t)} - \overline{\gamma} \right) \right] \geq 0 \right\}$$

for almost every $t \in [a,b]$ and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} \,\middle|\, \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \right\}$$

for a. e. $t \in [a,b]$

Also, we shall give the following lemma so as to prove the next inequality.

Lemma 2. [14] *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, the sets $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex, and closed convex sets and*

$$\overline{U}_{[a,]}(\gamma,\Gamma) = \overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Theorem 4. *Let $f : [a,b] \to \mathbb{C}$ be an $n-$time differentiable function on (a,b) and $x \in \left[a, \frac{a+b}{2}\right]$. If there exists $\gamma_i, \Gamma_i \in \mathbb{C}$ with $\gamma_i \neq \Gamma_i$, $i = 1,2,3$, such that*

$$f^{(n)} \in \overline{\Delta}_{[a,x]}(\gamma_1,\Gamma_1) \cap \overline{\Delta}_{[x,a+b-x]}(\gamma_2,\Gamma_2) \cap \overline{\Delta}_{[a+b-x,b]}(\gamma_3,\Gamma_3), \quad (5)$$

then we have the inequality

$$\left| S(f:n,x) - [1+(-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} - \right.$$
$$- \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(x-a)^{n+1}}{(n+1)!} +$$

$$+(-1)^n\int_a^b f(t)dt\Bigg| \le \frac{\varepsilon_1+\varepsilon_3}{2}\frac{(x-a)^{n+1}}{(n+1)\,!}+\frac{\varepsilon_2}{(n+1)\,!}\Big(\frac{a+b}{2}-x\Big)^{n+1} \quad (6)$$

where $\varepsilon_1 = |\Gamma_1 t(x)-\gamma_1(x)|\,,\ \varepsilon_2 = |\Gamma_2(x)-\gamma_2(x)|\,,\ \varepsilon_3 = |\Gamma_3(x)-\gamma_3(x)|\,.$

Proof. If we take absolute value of both left- and right-hand side of (3) for $\lambda_1(x) = \dfrac{\gamma_1(x)+\Gamma_1(x)}{2}$, $\lambda_2(x) = \dfrac{\gamma_2(x)+\Gamma_2(x)}{2}$, $\lambda_3(x) = \dfrac{\gamma_3(x)+\Gamma_3(x)}{2}$, we get the inequality

$$\Bigg|S(f:n,x)-[1+(-1)^n]\frac{\gamma_2(x)+\Gamma_2(x)}{2(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1}-$$
$$-\left[\frac{\gamma_1(x)+(-1)^n\gamma_3(x)+\Gamma_1(x)+(-1)^n\Gamma_3(x)}{2}\right]\frac{(x-a)^{n+1}}{(n+1)!}+$$
$$+(-1)^n\int_a^b f(t)dt\Bigg| \le \int_a^x \frac{(t-a)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_1(x)+\Gamma_1(x)}{2}\right|dt+$$
$$+\int_x^{a+b-x}\frac{1}{n!}\left|t-\frac{a+b}{2}\right|^n\left|f^{(n)}(t)-\frac{\gamma_2(x)+\Gamma_2(x)}{2}\right|dt+$$
$$+\int_{a+b-x}^b \frac{(b-t)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_3(x)+\Gamma_3(x)}{2}\right|dt.$$

Utilizing condition (5), on account of the definition of $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$, we write the inequality

$$\int_a^x \frac{(t-a)^n}{n!}\left|f^{(n)}(t)-\frac{\gamma_1(x)+\Gamma_1(x)}{2}\right|dt \le \frac{1}{2}\left|\Gamma_1(x)-\gamma_1(x)\right|\int_a^x\frac{(t-a)^n}{n!}dt =$$
$$=\frac{1}{2}\left|\Gamma_1(x)-\gamma_1(x)\right|\frac{(x-a)^{n+1}}{(n+1)!}.$$

Similarly, the results of the other integrals can also be obtained. Thus, the proof is completed. □

Corollary 1. *To get the following inequalities, we use:*

- *the Hölder inequality*

$$mn + pq \le (m^{\alpha} + p^{\alpha})^{\frac{1}{\alpha}} \left(n^{\beta} + q^{\beta}\right)^{\frac{1}{\beta}},$$

where $m, n, p, q \ge 0$ *and* $\alpha > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$;

- *the identity*

$$\max\{X, Y\} = \frac{X+Y}{2} + \left|\frac{X-Y}{2}\right|;$$

- *the maximum property of* $\max\{a^n, b^n\} = (\max\{a, b\})^n$ *for* $a, b > 0$ *and* $n \in \mathbb{N}$ *in the left-hand side of inequality* (6).

The obtained inequalities are

$$\left|S(f:n,x) - [1+(-1)^n]\frac{\gamma_2(x)+\Gamma_2(x)}{2(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1} - \right.$$

$$\left. - \left[\frac{\gamma_1(x)+(-1)^n\gamma_3(x)+\Gamma_1(x)+(-1)^n\Gamma_3(x)}{2}\right]\frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n\int\limits_a^b f(t)dt\right|$$

$$\le \begin{cases} \dfrac{1}{(n+1)!}\left[(x-a)^{n+1} + \left(\dfrac{a+b}{2}-x\right)^{n+1}\right]\max\left\{\dfrac{\varepsilon_1+\varepsilon_3}{2}, \varepsilon_2\right\}, \\ \dfrac{1}{t(n+1)!}\left[(x-a)^{(n+1)p} + \left(\dfrac{a+b}{2}-x\right)^{(n+1)p}\right]^{\frac{1}{p}}\left[\left(\dfrac{\varepsilon_1+\varepsilon_3}{2}\right)^q + \varepsilon_2^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{for } p > 1, \quad \frac{1}{p}+\frac{1}{q} = 1, \\ \dfrac{1}{(n+1)!}\left[\dfrac{b-a}{4} + |x - \dfrac{3a+b}{4}|\right]^{n+1}\left[\dfrac{\varepsilon_1+\varepsilon_3}{2} + \varepsilon_2\right], \end{cases}$$

where $\varepsilon_1 = |\Gamma_1(x) - \gamma_1(x)|$, $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$, $\varepsilon_3 = |\Gamma_3(x) - \gamma_3(x)|$.

Remark 1. *Let* f *and* x *be defined as in Theorem 4. If there exists* $\gamma_i, \Gamma_i \in \mathbb{C}$ *with* $\gamma_i \neq \Gamma_i$, $i = 1, 2$, *such that*

$$f^{(n)} \in \overline{\Delta}_{[a,x]}(\gamma_1, \Gamma_1) \cap \overline{\Delta}_{[x,a+b-x]}(\gamma_2, \Gamma_2) \cap \overline{\Delta}_{[a+b-x,b]}(\gamma_1, \Gamma_1),$$

then we have

$$\left|S(f:n,x) - [1+(-1)^n]\frac{\gamma_2(x)+\Gamma_2(x)}{2(n+1)!}\left(\frac{a+b}{2}-x\right)^{n+1} - \right.$$

$$-\frac{[1+(-1)^n][\gamma_1(x)+\Gamma_1(x)]}{2}\frac{(x-a)^{n+1}}{(n+1)!}+(-1)^n\int_a^b f(t)dt\Big| \leq$$

$$\leq \frac{1}{(n+1)!}\left[\varepsilon_1(x-a)^{n+1}+\varepsilon_2\Big(\frac{a+b}{2}-x\Big)^{n+1}\right] \quad (7)$$

where $\varepsilon_1 = |\Gamma_1(x)-\gamma_1(x)|$ *and* $\varepsilon_2 = |\Gamma_2(x)-\gamma_2(x)|$. *Additionally, if there exists* $\gamma_1, \Gamma_1 \in \mathbb{C}$ *with* $\gamma_1 \neq \Gamma_1$ *such that* $f^{(n)} \in \overline{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$, *then we have the conclusion*

$$\Big|S(f:n,x)+(-1)^n\int_a^b f(t)dt-$$

$$-\frac{[1+(-1)^n][\gamma_1(x)+\Gamma_1(x)]}{2(n+1)!}\left[\Big(\frac{a+b}{2}-x\Big)^{n+1}+(x-a)^{n+1}\right]\Big| \leq$$

$$\leq \frac{|\Gamma_1(x)-\gamma_1(x)|}{(n+1)!}\left[\Big(\frac{a+b}{2}-x\Big)^{n+1}+(x-a)^{n+1}\right]. \quad (8)$$

Remark 2. *If we select* $x=a$ *in inequality* (6), *we have*

$$\Big|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}\left[f^{(k)}(b)+(-1)^k f^{(k)}(a)\right]}{(k+1)!}\left[(-1)^k\Big(\frac{b-a}{2}\Big)^{k+1}\right]-$$

$$-[1+(-1)^n]\frac{\gamma_2(x)+\Gamma_2(x)}{2(n+1)!}\Big(\frac{a+b}{2}-x\Big)^{n+1}+(-1)^n\int_a^b f(t)dt\Big| \leq$$

$$\leq \frac{|\Gamma_2(x)-\gamma_2(x)|}{(n+1)!}\Big(\frac{b-a}{2}\Big)^{n+1}.$$

Remark 3. *If we take* $x=\dfrac{a+b}{2}$ *in the inequality* (6), *then one concludes the inequality*

$$\Big|\sum_{k=0}^{n-1}\frac{(-1)^{n+1}f^{(k)}(\frac{a+b}{2})[1+(-1)^k]}{(k+1)!}\Big(\frac{b-a}{2}\Big)^{k+1}+(-1)^n\int_a^b f(t)dt-$$

$$-\left[\frac{\gamma_1(x)+(-1)^n\gamma_3(x)+\Gamma_1(x)+(-1)^n\Gamma_3(x)}{2}\right]\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\Big| \leq$$

$$\leq \frac{|\Gamma_1(x) - \gamma_1(x)| + |\Gamma_3(x) - \gamma_3(x)|}{2(n+1)!} \left(\frac{b-a}{2}\right)^{n+1}.$$

Also, should we use the condition of the result (7) in this inequality, then we can find a new inequality.

Remark 4. *Substitution of* $x = \dfrac{3a+b}{4}$ *in (6) gives*

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[1+(-1)^k\right]\left[f^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)\right](b-a)^{k+1}}{4^{k+1}(k+1)!} - \right.$$

$$- \left[1+(-1)^n\right] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} + (-1)^n \int_a^b f(t)dt -$$

$$\left. - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2}\right] \frac{(b-a)^{n+1}}{4^{n+1}(n+1)!} \right| \leq$$

$$\leq \frac{1}{(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \left[\frac{\varepsilon_1 + \varepsilon_3}{2} + \varepsilon_2\right]$$

where $\varepsilon_1 = |\Gamma_1(x) - \gamma_1(x)|$, $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$ *and* $\varepsilon_3 = |\Gamma_3(x) - \gamma_3(x)|$. *What is more, applying the condition of the result (8) to this inequality, a new inequality can be found.*

In addition to these results, one can deduce some inequalities, taking $n = 1$ in inequality (6) or the other results related to (6); these inequalities were published by Dragomir [15]. Furthermore, if we take $n = 2$ in (6) or the other results connected to (6), then we obtain some inequalities presented in [23] that is published by Sarikaya et. al.

3. The case when $f^{(n)}$ is of Bounded Variation.

We begin with the definition of bounded-variation functions and the concept of total variation, which is used throughout this section.

Definition 1. *Let* $P : a = x_0 < x_1 < \ldots < x_n = b$ *be any partition of* $[a, b]$ *and let* $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$*; then* f *is said to be of bounded variation, if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum \Delta f\,(P)$ denote the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b (f) := \sup\left\{\sum \Delta f\,(P) : P \in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

Now, a perturbed inequality of the Ostrowski type for functions whose high-order derivatives are of bounded variation, are established in the following theorem.

Theorem 5. *Let $f : I \to \mathbb{C}$ be an n time differentiable function on I° and $[a,b] \subset I^\circ$. If the n-th derivative $f^{(n)}$ is of bounded variation on $[a,b]$, then we have*

$$\Bigg| S(f:n,x) + (-1)^n \int_a^b f(t)dt - \left[f^{(n)}(a) + (-1)^n f^{(n)}(b)\right] \frac{(x-a)^{n+1}}{(n+1)\,!} -$$
$$- \left[1 + (-1)^n\right] \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2\,(n+1)\,!} \left(\frac{a+b}{2} - x\right)^{n+1} \Bigg| \le$$
$$\le \frac{(x-a)^{n+1}}{t(n+1)!} \left[\bigvee_a^x (f^{(n)}) + \bigvee_{a+b-x}^b (f^{(n)})\right] +$$
$$+ \frac{1}{(n+1)!} \left(\frac{a+b}{2} - x\right)^{n+1} \bigvee_x^{a+b-x} (f^{(n)}) \quad (9)$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Writing $f^{(n)}(a)$, ${}^{(f^{(n)}(x) + f^{(n)}(a+b-x))}\!/_2$, $f^{(n)}(b)$ instead of $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$ in equation (3) respectively, then taking modulus of this equality, we find that

$$\Bigg| S(f:n,x) + (-1)^n \int_a^b f(t)dt - \left[f^{(n)}(a) + (-1)^n f^{(n)}(b)\right] \frac{(x-a)^{n+1}}{(n+1)\,!} -$$
$$- \left[1 + (-1)^n\right] \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2\,(n+1)\,!} \left(\frac{a+b}{2} - x\right)^{n+1} \Bigg| \le$$

$$\leq \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) - f^{(n)}(a) \right| dt +$$

$$+ \int_{a+b-x}^{b} \frac{(b-t)^n}{n!} \left| f^{(n)}(t) - f^{(n)}(b) \right| dt +$$

$$+ \int_x^{a+b-x} \frac{1}{n!} \left| t - \frac{a+b}{2} \right|^n \left| f^{(n)}(t) - \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2} \right| dt.$$

Noting that $f^{(n)} : I^\circ \to \mathbb{C}$ is of bounded variation on $[a, x]$, we get

$$\left| f^{(n)}(t) - f^{(n)}(a) \right| \leq \bigvee_a^x \left(f^{(n)} \right)$$

and observe that

$$\int_a^x \frac{(t-a)^n}{n!} dt = \frac{(x-a)^{n+1}}{(n+1)!}.$$

The other integrals are also examined by noting that $f^{(n)} : I^\circ \to \mathbb{C}$ is of bounded variation on $[x, a+b-x]$ and $[a+b-x, b]$: we can find the result (9), which finishes the proof. □

Remark 5. *Suppose that all assumptions of Theorem 5 hold. If we take $x = a$ in the inequality given this theorem, we have*

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} \left[f^{(k)}(b) + (-1)^k f^{(k)}(a) \right]}{(k+1)!} \left[(-1)^k \left(\frac{b-a}{2} \right)^{k+1} \right] - \right.$$

$$\left. - \left[1 + (-1)^n \right] \frac{f^{(n)}(a) + f^{(n)}(a+b-x)}{2(n+1)!} \left(\frac{b-a}{2} \right)^{n+1} + (-1)^n \int_a^b f(t) dt \right| \leq$$

$$\leq \frac{1}{(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} \bigvee_a^b \left(f^{(n)} \right).$$

In addition, if we choose $x = \frac{a+b}{2}$, we get the midpoint inequality

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} f^{(k)} \left(\frac{a+b}{2} \right) \left[1 + (-1)^k \right]}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} - \right.$$

$$- \left[f^{(n)}(a) + (-1)^n f^{(n)}(b)\right] \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} + -1)^n \int_a^b f(t)dt \Bigg| \leq$$

$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} \bigvee_a^b \left(f^{(n)}\right).$$

Finally, should we take $x = \dfrac{3a+b}{4}$, *we have*

$$\Bigg| \sum_{k=0}^{n-1} \frac{(-1)^{n+1}\left[1+(-1)^k\right]\left[f^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)\right](b-a)^{k+1}}{4^{k+1}(k+1)!} +$$

$$+ (-1)^n \int_a^b f(t)dt - \left[f^{(n)}(a) + (-1)^n f^{(n)}(b)\right] \frac{(b-a)^{n+1}}{4^{n+1}(n+1)!} -$$

$$- [1+(-1)^n] \frac{f^{(n)}\left(\frac{3a+b}{4}\right) + f^{(n)}\left(\frac{a+3b}{4}\right)}{2(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \Bigg| \leq$$

$$\leq \frac{1}{(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \bigvee_a^b t(f^{(n)}).$$

Besides the results that are presented in this section, taking $n = 1$ in the inequality (9) or the other results pertaining to (6), we obtain some inequalities given in [15] by Dragomir. What is more, should we take $n = 2$ in expression (6) or the other results interested in (6), we can find some inequalities presented in [23] by Sarikaya et. al.

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