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SMIRNOV'S INEQUALITY FOR POLYNOMIALS HAVING ZEROS OUTSIDE THE UNIT DISC

Abstract. In 1887, the famous chemist D. I. Mendeleev posed the following problem: to estimate $|f'(x)|$ for a real polynomial $f(x)$, satisfying the condition $|f(x)| \leq M$ on $[a, b]$. This question arose when Mendeleev was studying aqueous solutions. The problem was solved by the famous mathematician A. A. Markov, and over the following 100 years was repeatedly modified and extended. For complex polynomials, important inequalities were obtained by S. N. Bernstein and V. I. Smirnov. Many other well-known mathematicians, such as Ch. Pommerenke, G. Szegő, Q. I. Rahman, G. Schmeisser, worked in this subject. Almost all results in this direction significantly use the following condition: all zeros of a majorizing polynomial belong to the closed unit disc. In this paper, we remove this condition. Here a majorizing polynomial may have zeros outside the unit disc. This allows to extend the inequalities of Bernstein and Smirnov.

Key words: *polynomial, the Smirnov inequality, the Bernstein inequality*

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1. Introduction. The subject discussed in this article can be traced back to the problem posed by the famous chemist D. I. Mendeleev in 1887: for a polynomial f , $\deg f = 2$, with $|f(x)| \leq M$ for $x \in [a, b]$, give an estimate for $|f'(x)|$ on $[a, b]$ [12, § 86]. Over the following 100 years, the Mendeleev problem was repeatedly modified and extended. For example, in the book by V. I. Smirnov and N. A. Lebedev [16, p. 340], this problem was formulated in a more general form:

Let $B \subset \mathbb{C}$ be a compact set, $f(z)$ be a polynomial with $\deg f = n \geq 1$, and suppose that $|f(z)| \leq M$ for all $z \in B$. Give an estimate for $|f'(z)|$ for $z \in B$.

The original problem posed by Mendelev [12, § 86] was solved by A. A. Markov in 1889 for a polynomial f with $\deg f = n$.

Theorem A. [9], [10, p. 51–75]. *Let f be a polynomial with $\deg f = n$ and suppose that $|f(x)| \leq M$ on $[-1, 1]$. Then, for $x \in [-1, 1]$:*

$$|f'(x)| \leq n^2 M. \quad (1)$$

Equality is attained only for the functions

$$f(x) = \pm MT_n(x)$$

where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials.

Inequality (1) was refined by S. N. Bernstein in 1912 in the following way:

Theorem B. [1, p. 25]. *Let f be a polynomial from Theorem A. Then, for $x \in [-1, 1]$*

$$|f'(x)| \leq \frac{nM}{\sqrt{1-x^2}}.$$

Theorem B was of a considerable use for answering the following question of the best approximation: is it possible to approximate the function $|x|$ by polynomials of degree n with an error $o(1/n)$ as $n \rightarrow \infty$? This question was posted in 1903 as a competition by the Belgian Academy of Science on a suggestion by Ch. J. de Vallée-Poussin. S. N. Bernstein gave the negative answer to this question in [1], which was published by the Belgian Academy as the prize-winning treatise. For more mathematical and historical details see, for example, [6, p. 157–161], [17, p. 182–186].

V. A. Markov obtained a result analogous to Theorem A for the k -th derivative of a polynomial f , $1 \leq k \leq n$, [11]. Note that Mendelev and Markovs dealt only with real polynomials. For more details concerning this question, we also refer the reader to the historical survey in [7].

In this paper, we consider complex polynomials. Let \mathbb{D} denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. V. I. Smirnov found the following solution to the Mendelev problem for complex polynomials with $B = \partial\mathbb{D}$:

Theorem C. [15], [16, ch.V, § 1, 2⁰, p. 346] *Let f be a polynomial with $\deg f = n$ and suppose that $|f(z)| \leq M$ on $\partial\mathbb{D}$. Then, for $z \in \partial\mathbb{D}$:*

$$|f'(z)| \leq Mn.$$

Equality holds only if $f(z) = e^{i\gamma} z^n$, $\gamma \in \mathbb{R}$.

Similar results have been obtained for some other classes of compact sets $B \subset \mathbb{C}$, for example, by Ch. Pommerenke [13] and G. Szegő [18] (see also [16, pp. 351, 352]). We shall not dwell on these results, since we are interested in the case $B = \partial\mathbb{D}$ only.

It is convenient to rewrite Theorem C as follows:

Theorem C'. Let f be a polynomial with $\deg f = n$ and suppose that $|f(z)| \leq |Mz^n|$ on $\partial\mathbb{D}$. Then

$$|f'(z)| \leq |(Mz^n)'|, \quad z \in \partial\mathbb{D}.$$

Equality holds only if $f(z) = e^{i\gamma} z^n$, $\gamma \in \mathbb{R}$.

In 1930, S. N. Bernstein obtained a generalization of Theorem C', replacing the polynomial Mz^n with an arbitrary polynomial F of degree n , and obtained the following result:

Theorem D. [2] (see also [3, p. 497], [14, p. 510]) Let f and F be polynomials, such that:

- 1) $\deg f \leq \deg F = n$,
- 2) $|f(z)| \leq |F(z)|$ on $\partial\mathbb{D}$,
- 3) F has all its zeros in $\overline{\mathbb{D}}$.

Then

$$|f'(z)| \leq |F'(z)| \quad \text{for } z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \quad (2)$$

For $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, equality holds only for $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

Let $R \geq 1$; denote by Ω_R the image of the disc $\{z \in \mathbb{C} : |z| \leq R\}$ under the mapping $\psi(z) = t(1+t)^{-1}$. For a polynomial f of degree at most n put

$$S_\alpha[f](z) = zf'(z) - n\alpha f(z),$$

where α is a complex constant.

V. I. Smirnov gave the following generalization of Theorem D:

Theorem E. [16, ch. V, § 1, p. 356] Let $R \geq 1$, f and F be polynomials from Theorem D. Then, for $|z| = R$,

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)| \quad (3)$$

for all $\alpha \in \Omega_R$.

For $\alpha \in \text{int } \Omega_R$ and $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, equality in (3) holds only for $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

From the maximum modulus principle it follows that it is possible to replace condition 2) in Theorem D and Theorem E by the equivalent condition: $|f(z)| \leq |F(z)|$ for $z \in \mathbb{C} \setminus \mathbb{D}$.

Putting $\alpha = 0$ in Theorem E, we obtain Theorem D. From Theorem D and Theorem E, we see that the differentiation operator and the Smirnov differential operator S_α preserve inequality between polynomials. Saying this, we mean the following: the inequality $|f(z)| \leq |F(z)|$ on $\partial\mathbb{D}$ remains valid if we apply one of these operators to polynomials f and F , with the conditions of Theorem D satisfied. Such type of operators was widely investigated (see, for example, [14], [4], [5], [19] and references therein). Almost all results concerning such operators significantly use condition 3) from Theorem D. We waived this condition allowing the polynomial F to have one zero outside the unit disc (see [8]): we proved the following

Theorem F. [8] Let $R \geq 1$. Let f and F be polynomials, such that

- 1) $\deg f \leq \deg F = n$,
- 2) $|f(z)| \leq |F(z)|$ for $z \in \mathbb{C} \setminus \mathbb{D}$,
- 3) z_0 is a unique zero of F lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$, k is order of z_0 , $1 \leq k \leq n-1$.

Then

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)| \quad (4)$$

for $|z| = R$ and for $\alpha \in D_R$, where D_R is one of the sets

a) complement to the disc

$$\left\{ \alpha \in \mathbb{C} : \left| \alpha - \frac{n-k}{n} \frac{R^2}{R^2-1} - \frac{R^2}{n} \frac{k}{R^2-|z_0|^2} \right| < \frac{n-k}{n} \frac{R}{R^2-1} + \frac{R}{n} \frac{k|z_0|}{|R^2-|z_0|^2|} \right\}$$

if $R > 1$ and $|z_0| \neq R$;

b) complement to the strip

$$\left\{ \alpha \in \mathbb{C} : \left| \text{Re } \alpha - \frac{n-k}{n} \frac{R^2}{R^2-1} - \frac{k}{2n} \right| < \frac{n-k}{n} \frac{R}{R^2-1} \right\}$$

if $R > 1$, $|z_0| = R$;

c) the half-plane

$$\left\{ \alpha \in \mathbb{C} : \text{Re } \alpha \leq \frac{k}{n} \left(\frac{1}{1-|z_0|} + \frac{1}{2} \left(1 - \frac{k}{n} \right) \right) \right\}$$

if $R = 1$.

For $z = z_0$, $k > 1$, equality in (4) holds for every $\alpha \in \mathbb{C}$ and every f and F satisfying conditions of Theorem F.

For the case $z = z_0$, $k = 1$, $\alpha \in \mathbb{C}$ and the case $z \neq z_0$, $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $\alpha \in \text{int } D_R$, in (4) equality holds only if $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

In this paper, we consider the case when the polynomial F has an arbitrary number of zeros outside the closed unit disc.

2. Refinement of the Smirnov inequality.

Theorem 1. Suppose that $R \geq 1$. Let f and F be polynomials, such that

- 1) $\deg f \leq \deg F = n$,
- 2) $|f(z)| \leq |F(z)|$ for $z \in \mathbb{C} \setminus \mathbb{D}$,
- 3) z_1, \dots, z_p are all the zeros of F lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$; k_1, \dots, k_p are their orders, correspondingly, $1 \leq k = k_1 + \dots + k_p \leq n - 1$.

Then

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)| \quad (5)$$

for $|z| = R$ and for $\alpha \in D(R, p, n)$, where $D(R, p, n)$ is one of the sets:

a) complement to the disc

$$\{\alpha \in \mathbb{C} : |\alpha - c_p| < r_p\}$$

where

$$c_p = \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{R^2}{n} \sum_{j=1}^p \frac{k_j}{R^2 - |z_j|^2},$$

$$r_p = \frac{n-k}{n} \frac{R}{R^2-1} + \frac{R}{n} \sum_{j=1}^p \frac{k_j |z_j|}{|R^2 - |z_j|^2|},$$

if $R > 1$ and all the zeros z_1, \dots, z_p do not belong to the circle $|z| = R$;

b) complement to the strip

$$\left\{ \alpha \in \mathbb{C} : |\operatorname{Re} \alpha - s_p| < \frac{n-k}{n} \frac{R}{R^2-1} \right\},$$

where

$$s_p = \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{k}{2n},$$

if $R > 1$ and $|z_1| = \dots = |z_p| = R$;

c) complement to the strip

$$\{\alpha \in \mathbb{C} : |\operatorname{Re} \alpha - x_{q,p}| < \rho_{q,p}\},$$

where

$$x_{q,p} = \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{R^2}{n} \sum_{j=1}^q \frac{k_j}{R^2 - |z_j|^2} + \frac{1}{2n} \sum_{j=q+1}^p k_j,$$

$$\rho_{q,p} = \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{R}{n} \sum_{j=1}^q \frac{k_j |z_j|}{|R^2 - |z_j|^2|},$$

if $R > 1$, $|z_1|, \dots, |z_q| \neq R$; $|z_{q+1}| = \dots = |z_p| = R$; $q \in \mathbb{N}$, $1 \leq k_1 + \dots + k_q \leq n-2$;

d) the half-plane

$$\left\{ \alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq \frac{1}{n} \sum_{j=1}^p \frac{k_j}{1 - |z_j|} + \frac{1}{2} \left(1 - \frac{k}{n} \right) \right\}$$

if $R = 1$.

For $z = z_j$, $j = 1, \dots, p$, $k_j > 1$ in (5), equality holds for every $\alpha \in \mathbb{C}$ and all pairs f and F satisfying the theorem conditions.

If $z = z_j$, $j = 1, \dots, p$, $k_j = 1$, $\alpha \in \mathbb{C}$ or $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $z \neq z_j$, $j = 1, \dots, p$, $\alpha \in \operatorname{int} D(R, p, n)$, then equality in (5) takes place only if $f = e^{i\gamma} F$, $\gamma \in \mathbb{R}$.

Remark. From the proof of Theorem 1, we see that item a) does not describe the whole set of values of α , for which (5) is true. The set $D(R, p, n)$ from item a) can be extended, but this extension requires much more complicated description.

Proof. By condition 3) of Theorem 1, we see that the polynomial F can be written in the form

$$F(z) = (z - z_1)^{k_1} \dots (z - z_p)^{k_p} F_1(z); \quad (6)$$

here $F_1(z) \neq 0$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$. By condition 2), the polynomial f has the representation

$$f(z) = (z - z_1)^{k_1} \dots (z - z_p)^{k_p} f_1(z). \quad (7)$$

We prove items a), b), and d) of Theorem 1 by mathematical induction. For $p = 1$, we obtain the statement of Theorem F.

Suppose that the theorem statement is true for $p - 1$, $1 \leq k_1 + \dots + k_{p-1} \leq n - 2$. Prove that Theorem 1 is valid for p , $2 \leq k = k_1 + \dots + k_p \leq n - 1$.

Denote (see (6) and (7))

$$F_2(z) = (z - z_1)^{k_1} \dots (z - z_{p-1})^{k_{p-1}} F_1(z),$$

$$f_2(z) = (z - z_1)^{k_1} \dots (z - z_{p-1})^{k_{p-1}} f_1(z).$$

Then $F(z) = (z - z_p)^{k_p} F_2(z)$, $f(z) = (z - z_p)^{k_p} f_2(z)$, and, by item 2) of Theorem 1, $|f_2(z)| \leq |F_2(z)|$, $|z| \geq 1$.

Let us find the set of constants α , such that (5) takes place for all z on the circle $|z| = R$. Write (5) as

$$|zf'(z) - \alpha n f(z)| \leq |zF'(z) - \alpha n F(z)|, \quad |z| = R,$$

or

$$\begin{aligned} & \left| z \left(k_p (z - z_p)^{k_p-1} f_2(z) + (z - z_p)^{k_p} f_2'(z) \right) - \alpha n (z - z_p)^{k_p} f_2(z) \right| \leq \\ & \leq \left| z \left(k_p (z - z_p)^{k_p-1} F_2(z) + (z - z_p)^{k_p} F_2'(z) \right) - \alpha n (z - z_p)^{k_p} F_2(z) \right|. \end{aligned} \quad (8)$$

For $z = z_p$, (8) takes place for all $\alpha \in \mathbb{C}$. If $z = z_p$ and $k_p > 1$, then equality in (8) holds for every f_2 and F_2 . Therefore, equality in (5) takes place for every pair of the polynomials f and F satisfying the conditions of the theorem.

If $z = z_p$ and $k_p = 1$, then, by (8), equality in (5) holds only if $f(z) = e^{i\gamma} F(z)$, $\gamma \in \mathbb{R}$.

Obviously, we can change the index p with any j , $j = 1, \dots, p - 1$, arguing as above about equality in (5).

Assuming, further, $z \neq z_p$, we rewrite (8) in the form

$$\left| zf_2'(z) - \left(\alpha n - k_p \frac{z}{z - z_p} \right) f_2(z) \right| \leq \left| zF_2'(z) - \left(\alpha n - k_p \frac{z}{z - z_p} \right) F_2(z) \right|, \quad |z| = R. \quad (9)$$

By the induction assumption, Theorem 1 is true for the polynomials f_2 and F_2 with n replaced by $n - k_p$. Hence, we can rewrite (9) as

$$|S_\beta[f_2](z)| \leq |S_\beta[F_2](z)|, \quad |z| = R, \quad (10)$$

where

$$\beta = \frac{1}{n - k_p} \left(\alpha n - k_p \frac{z}{z - z_p} \right) \in D(R, p - 1, n - k_p). \quad (11)$$

Then

$$\alpha = \left(1 - \frac{k_p}{n}\right)\beta + \frac{k_p}{n} \frac{z}{z - z_p}, \quad |z| = R, \quad (12)$$

and for a fixed z , $|z| = R$, the set G_z of values of α is the set $\left(1 - \frac{k_p}{n}\right)D(R, p-1, n-k_p)$ shifted by $\frac{k_p}{n} \frac{z}{z - z_p}$. Therefore, the set of the values α , such that (5) is true, is the set $\bigcap_{|z|=R} G_z$.

First, we consider the case $\mathbf{R} > 1$.

a) Let $|\mathbf{z}_j| \neq \mathbf{R}$ for all $\mathbf{j} = 1, \dots, \mathbf{p}$. By the inductive assumption, inequality (10) is true for β that belong to $D(R, p-1, n-k_p)$. This set is the complement to the open disc

$$C = \{\beta \in \mathbb{C} : |\beta - c_{p-1}| < r_{p-1}\},$$

where

$$c_{p-1} = \frac{(n - k_p) - k_1 - \dots - k_{p-1}}{n - k_p} \frac{R^2}{R^2 - 1} + \frac{R^2}{n - k_p} \sum_{j=1}^{p-1} \frac{k_j}{R^2 - |z_j|^2},$$

$$r_{p-1} = \frac{(n - k_p) - k_1 - \dots - k_{p-1}}{n - k_p} \frac{R}{R^2 - 1} + \frac{R}{n - k_p} \sum_{j=1}^{p-1} \frac{k_j |z_j|}{|R^2 - |z_j|^2|}.$$

The function $\frac{z}{z - z_p}$ maps $|z| = R$ to the circle with center $\frac{R^2}{R^2 - |z_p|^2}$ and radius $\frac{R|z_p|}{|R^2 - |z_p|^2|}$. Hence, the set of all centers of the discs

$$C_z = \left(1 - \frac{k_p}{n}\right)C + \frac{k_p}{n} \frac{z}{z - z_p}, \quad |z| = R,$$

is the circle with center

$$\begin{aligned} & \left(1 - \frac{k_p}{n}\right) c_{p-1} + \frac{k_p}{n} \frac{R^2}{R^2 - |z_p|^2} = \\ &= \frac{n - k_p}{n} \left(\frac{n - k}{n - k_p} \frac{R^2}{R^2 - 1} + \frac{R^2}{n - k_p} \sum_{j=1}^{p-1} \frac{k_j}{R^2 - |z_j|^2} \right) + \frac{k_p}{n} \frac{R^2}{R^2 - |z_p|^2} = \\ &= \frac{n - k}{n} \frac{R^2}{R^2 - 1} + \frac{R^2}{n} \sum_{j=1}^p \frac{k_j}{R^2 - |z_j|^2} = c_p \end{aligned}$$

and radius $\frac{k_p}{n} \frac{R|z_p|}{|R^2 - |z_p|^2|}$. Note that

$$\bigcap_{|z|=R} G_z = \bigcap_{|z|=R} (\mathbb{C} \setminus C_z)$$

contains the complement to the disc

$$\{\alpha \in \mathbb{C} : |\alpha - c_p| < A\},$$

where

$$\begin{aligned} A &= \frac{k_p}{n} \frac{R|z_p|}{|R^2 - |z_p|^2|} + \left(1 - \frac{k_p}{n}\right) r_{p-1} = \\ &= \frac{k_p}{n} \frac{R|z_p|}{|R^2 - |z_p|^2|} + \frac{n - k_p}{n} \left(\frac{n - k}{n - k_p} \frac{R}{R^2 - 1} + \frac{R}{n - k_p} \sum_{j=1}^{p-1} \frac{k_j |z_j|}{|R^2 - |z_j|^2|} \right) = \\ &= \frac{n - k}{n} \frac{R}{R^2 - 1} + \frac{R}{n} \sum_{j=1}^p \frac{k_j |z_j|}{|R^2 - |z_j|^2|} = r_p. \end{aligned}$$

Hence, $D(R, p, n) \subset \bigcap_{|z|=R} G_z$ and a) is proved.

b) Let $|z_j| = R$ for all $j = 1, \dots, p$. By the induction assumption, β belongs to the set

$$D(R, p-1, n-k_p) = \left\{ \beta \in \mathbb{C} : |\operatorname{Re} \beta - s_{p-1}| \geq \frac{(n-k_p) - k_1 - \dots - k_{p-1}}{n - k_p} \frac{R}{R^2 - 1} \right\},$$

where

$$s_{p-1} = \frac{(n - k_p) - k_1 - \dots - k_{p-1}}{n - k_p} \frac{R^2}{R^2 - 1} + \frac{k_1 + \dots + k_{p-1}}{2(n - k_p)}.$$

The function $\frac{z}{z - z_p}$ maps the circle $|z| = R$ to the straight line $\operatorname{Re} z = \frac{1}{2}$.

Therefore, in this case we have

$$D(R, p, n) = \bigcap_{|z|=R} G_z = \left(1 - \frac{k_p}{n}\right) D(R, p-1, n-k_p) + \frac{k_p}{2n}.$$

This implies that $D(R, p, n) = \{\alpha \in \mathbb{C} : |\operatorname{Re} \alpha - B| \geq D\}$, where

$$B = \left(1 - \frac{k_p}{n}\right) s_{p-1} + \frac{k_p}{2n} =$$

$$= \frac{n-k_p}{n} \left(\frac{n-k}{n-k_p} \frac{R^2}{R^2-1} + \frac{k_1 + \dots + k_{p-1}}{2(n-k_p)} \right) + \frac{k_p}{2n} = \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{k}{2n} = s_p$$

and

$$D = \left(1 - \frac{k_p}{n}\right) \frac{n-k}{n-k_p} \frac{R}{R^2-1} = \frac{n-k}{n} \frac{R}{R^2-1}.$$

d) Consider the case $\mathbf{R} = \mathbf{1}$. By the induction assumption, β belongs to the half-plane

$$D(R, p-1, n-k_p) = \left\{ \beta \in \mathbb{C} : \operatorname{Re} \beta \leq \frac{1}{n-k_p} \sum_{j=1}^{p-1} \frac{k_j}{1-|z_j|} + \frac{1}{2} \left(1 - \frac{k_1 + \dots + k_{p-1}}{n-k_p} \right) \right\}.$$

Since the function $\frac{z}{z-z_p}$ maps $\partial\mathbb{D}$ to the circle with diameter $\left[\frac{1}{1-|z_p|}, \frac{1}{1+|z_p|} \right]$, we learn that the parameter α (see (12)) belongs to the set

$$D(1, p, n) = \bigcap_{|z|=1} G_z,$$

where

$$G_z = \left(1 - \frac{k_p}{n}\right) D(1, p-1, n-k_p) + \frac{k_p}{n} \frac{z}{z-z_p}.$$

This intersection is the set $D(1, p, n) = \{\alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq E\}$, where

$$\begin{aligned} E &= \left(1 - \frac{k_p}{n}\right) \left[\frac{1}{n-k_p} \sum_{j=1}^{p-1} \frac{k_j}{1-|z_j|} + \frac{1}{2} \left(1 - \frac{k_1 + \dots + k_{p-1}}{n-k_p} \right) \right] + \frac{k_p}{n} \frac{1}{1-|z_p|} = \\ &= \frac{1}{n} \sum_{j=1}^p \frac{k_j}{1-|z_j|} + \frac{1}{2} \left(1 - \frac{k}{n} \right). \end{aligned}$$

Let us consider separately the case c) when $\mathbf{R} > \mathbf{1}$, $\mathbf{z}_1, \dots, \mathbf{z}_q$ **do not lie on the circle** $|\mathbf{z}| = \mathbf{R}$, but $|\mathbf{z}_{q+1}| = \dots = |\mathbf{z}_p| = \mathbf{R}$, $1 \leq k_1 + \dots + k_q \leq n-2$, $q \in \mathbb{N}$.

Apply the proved item a) of Theorem 1 to the polynomials

$$\tilde{F}_q(z) = (z-z_1)^{k_1} \dots (z-z_q)^{k_q} F_1(z), \quad \tilde{f}_q(z) = (z-z_1)^{k_1} \dots (z-z_q)^{k_q} f_1(z),$$

where f_1 and F_1 are the polynomials from representations (6) and (7). We obtain

$$|S_{\alpha_q}[\tilde{f}_q](z)| \leq |S_{\alpha_q}[\tilde{F}_q](z)| \quad (13)$$

for $|z| = R$ and α_q from the set $D(R, q, n - k_{q+1} - \dots - k_p)$. This set is the complement to the disc $\{\alpha_q \in \mathbb{C} : |\alpha_q - c_q| < r_q\}$. Consider the polynomials

$$\tilde{F}_{q+1}(z) = (z - z_{q+1})^{k_{q+1}} \tilde{F}_q(z), \quad \tilde{f}_{q+1}(z) = (z - z_{q+1})^{k_{q+1}} \tilde{f}_q(z).$$

Similarly to how we previously rewrote inequality (5) in the form (10), we rewrite the inequality

$$|S_{\alpha_{q+1}}[\tilde{f}_{q+1}](z)| \leq |S_{\alpha_{q+1}}[\tilde{F}_{q+1}](z)|, \quad |z| = R,$$

in form (13), changing n to $n - k_{q+2} - \dots - k_p$. Similarly to (12), the parameters α_{q+1} and α_q are connected by the following relationship:

$$\alpha_{q+1} = \left(1 - \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p}\right) \alpha_q + \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p} \frac{z}{z - z_{q+1}}.$$

Since the function $\frac{z}{z - z_{q+1}}$ maps the circle $|z| = R$ to the straight line

$\operatorname{Re} z = \frac{1}{2}$ and $\alpha_q \in D(R, q, n - k_{q+1} - \dots - k_p)$, we learn that α_{q+1} belongs to the set

$$\{\alpha_{q+1} \in \mathbb{C} : |\operatorname{Re} \alpha_{q+1} - H| \geq I\},$$

where

$$\begin{aligned} H &= \left(1 - \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p}\right) c_q + \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p} \frac{1}{2} = \\ &= \frac{n - k_{q+1} - \dots - k_p}{n - k_{q+2} - \dots - k_p} \left(\frac{(n - k_{q+1} - \dots - k_p) - k_1 - \dots - k_q}{n - k_{q+1} - \dots - k_p} \frac{R^2}{R^2 - 1} + \right. \\ &\quad \left. + \frac{R^2}{n - k_{q+1} - \dots - k_p} \sum_{j=1}^q \frac{k_j}{R^2 - |z_j|^2} \right) + \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p} \frac{1}{2} = \\ &= \frac{n - k}{n - k_{q+2} - \dots - k_p} \frac{R^2}{R^2 - 1} + \\ &\quad + \frac{R^2}{n - k_{q+2} - \dots - k_p} \sum_{j=1}^q \frac{k_j}{R^2 - |z_j|^2} + \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p} \frac{1}{2} = x_{q, q+1} \end{aligned}$$

and

$$\begin{aligned}
 I &= \left(1 - \frac{k_{q+1}}{n - k_{q+2} - \dots - k_p}\right) r_q = \\
 &= \frac{n - k_{q+1} - \dots - k_p}{n - k_{q+2} - \dots - k_p} \left(\frac{(n - k_{q+1} - \dots - k_p) - k_1 - \dots - k_q}{n - k_{q+1} - \dots - k_p} \frac{R}{R^2 - 1} + \right. \\
 &\quad \left. + \frac{R}{n - k_{q+1} - \dots - k_p} \sum_{j=1}^q \frac{k_j |z_j|}{|R^2 - |z_j|^2|} \right) = \\
 &= \frac{n - k}{n - k_{q+2} - \dots - k_p} \frac{R}{R^2 - 1} + \frac{R}{n - k_{q+2} - \dots - k_p} \sum_{j=1}^q \frac{k_j |z_j|}{|R^2 - |z_j|^2|} = \rho_{q, q+1}
 \end{aligned}$$

in accordance with the definition of $\rho_{q, j}$ from the statement of Theorem 1. Therefore, we have proved item c) of Theorem 1 in the case when only one zero belongs to the circle $|z| = R$, i. e., $p = q + 1$.

Further, we prove c) by mathematical induction. Suppose that c) takes place for the polynomials

$$\tilde{F}_{p-1}(z) = (z - z_{p-1})^{k_{p-1}} \dots (z - z_1)^{k_1} F_1(z),$$

$$\tilde{f}_{p-1}(z) = (z - z_{p-1})^{k_{p-1}} \dots (z - z_1)^{k_1} f_1(z).$$

Prove that c) is true for the polynomials

$$F(z) = (z - z_p)^{k_p} \tilde{F}_{p-1}(z), \quad f(z) = (z - z_p)^{k_p} \tilde{f}_{p-1}(z).$$

As above, we rewrite (5) in the form

$$|S_{\alpha_{p-1}}[\tilde{f}_{p-1}](z)| \leq |S_{\alpha_{p-1}}[\tilde{F}_{p-1}](z)|, \quad |z| = R,$$

with

$$\alpha = \left(1 - \frac{k_p}{n}\right) \alpha_{p-1} + \frac{k_p}{n} \frac{z}{z - z_p}, \quad |z| = R,$$

see (8)–(12) for details.

The function $\frac{z}{z - z_p}$ maps the circle $|z| = R$ to the straight line $\operatorname{Re} z = \frac{1}{2}$.

By the assumption,

$$\alpha_{p-1} \in D(R, p-1, n - k_p) = \{\alpha_{p-1} \in \mathbb{C} : |\operatorname{Re} \alpha_{p-1} - x_{q, p-1}| \geq \rho_{q, p-1}\}.$$

Therefore, α belongs to the set

$$\left(1 - \frac{k_p}{n}\right) D(R, p-1, n-k_p) + \frac{k_p}{2n}$$

that is the set $\{\alpha \in \mathbb{C} : |\operatorname{Re} \alpha - J| \geq K\}$, where

$$\begin{aligned} J &= \left(1 - \frac{k_p}{n}\right) x_{q,p-1} + \frac{k_p}{2n} = \\ &= \frac{n-k_p}{n} \left(\frac{(n-k_p) - k_1 - \dots - k_{p-1}}{n-k_p} \frac{R^2}{R^2-1} + \right. \\ &\quad \left. + \frac{R^2}{n-k_p} \sum_{j=1}^q \frac{k_j}{R^2 - |z_j|^2} + \frac{1}{2(n-k_p)} \sum_{j=q+1}^{p-1} k_j \right) + \frac{k_p}{2n} = \\ &= \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{R^2}{n} \sum_{j=1}^q \frac{k_j}{R^2 - |z_j|^2} + \frac{1}{2n} \sum_{j=q+1}^p k_j = x_{q,p} \end{aligned}$$

and

$$\begin{aligned} K &= \left(1 - \frac{k_p}{n}\right) \rho_{q,p-1} = \\ &= \frac{n-k_p}{n} \left(\frac{(n-k_p) - k_1 - \dots - k_{p-1}}{n-k_p} \frac{R^2}{R^2-1} \frac{R}{n-k_p} \sum_{j=1}^q \frac{k_j |z_j|}{|R^2 - |z_j|^2|} \right) = \\ &= \frac{n-k}{n} \frac{R^2}{R^2-1} + \frac{R}{n} \sum_{j=1}^q \frac{k_j |z_j|}{|R^2 - |z_j|^2|} = \rho_{q,p}. \end{aligned}$$

To prove Theorem 1, it remains to establish when (5) turns into equality for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $z \neq z_j$, $j = 1, \dots, p$, $\alpha \in \operatorname{int} D(R, p, n)$. For $p = 1$, see Theorem F. Suppose that the theorem statement is true for $p-1$, $1 \leq k_1 + \dots + k_{p-1} \leq n-2$. For the mentioned z and $\alpha \in \operatorname{int} D(R, p, n)$, (5) turns to equality if and only if (10) turns into equality for this z and $\beta \in \operatorname{int} D(R, p-1, n-k_p)$. By the assumption, it is possible only in the case $f_2 = e^{i\gamma} F_2$, $\gamma \in \mathbb{R}$. Thus, $f = e^{i\gamma} F$. \square

3. The Bernstein inequality inside the unit disc. As we have seen above, the classical Bernstein inequality (Theorem D) can be obtained from the Smirnov inequality (Theorem E), taking $\alpha = 0$. However, we can not apply this reasoning to the polynomials with zeros in $\mathbb{C} \setminus \overline{\mathbb{D}}$, because

the domains $D(R, p, n)$ from Theorem 1 do not always contain the point $\alpha = 0$. Consequently, our first aim is to find a condition to guarantee that the point $\alpha = 0$ belongs to the domain $D(R, p, n)$.

The case when the polynomial F from Theorem 1 has a unique zero outside \mathbb{D} was considered in [8]. The following lemma was proved:

Lemma 1. [8] *If the polynomial F from Theorem 1 has a unique zero z_0 of order k in $\mathbb{C} \setminus \overline{\mathbb{D}}$, $1 < R < |z_0|$, then the set $D(R, 1, n)$ contains the point $\alpha = 0$ if and only if R does not belong to the interval $\left(\left(1 - \frac{k}{n}\right)|z_0| - \frac{k}{n}; \left(1 - \frac{k}{n}\right)|z_0| + \frac{k}{n}\right)$.*

Now we consider the case when F has an arbitrary number of zeros outside \mathbb{D} .

Lemma 2. *Let F be the polynomial from Theorem 1, z_1, \dots, z_p be all the zeros of F in $\mathbb{C} \setminus \overline{\mathbb{D}}$, $1 < R < \min_{j=1, \dots, p} |z_j| = |z_p|$, k_1, \dots, k_p be orders of these zeros, correspondingly, $k = k_1 + \dots + k_p$. If*

$$R \leq \left(1 - \frac{k}{n}\right)|z_p| - \frac{k}{n}, \quad (14)$$

then $0 \in D(R, p, n)$, where $D(R, p, n)$ is the domain from Theorem 1.

Proof. Since $|z_j| > R > 1$ for $j = 1, \dots, p$, we see that $D(R, p, n)$ is the domain from item a) of Theorem 1. For $0 \in D(R, p, n)$, it is sufficient to prove that $|c_p| \geq r_p$, where c_p and r_p are the constants from item a) of Theorem 1.

Prove that $c_p > 0$ under conditions of Lemma 2, i. e.,

$$\frac{n - k}{n} \frac{R^2}{R^2 - 1} > \frac{R^2}{n} \sum_{j=1}^p \frac{k_j}{|z_j|^2 - R^2}$$

or

$$\frac{n - k}{R^2 - 1} > \sum_{j=1}^p \frac{k_j}{|z_j|^2 - R^2}.$$

The last inequality follows from the inequality

$$\frac{n - k}{R^2 - 1} > \sum_{j=1}^p \frac{k_j}{|z_p|^2 - R^2} = \frac{k}{|z_p|^2 - R^2}. \quad (15)$$

Inequality (15) is equivalent to

$$n - k > \frac{R^2 - 1}{|z_p|^2 - R^2} k = \left(\frac{|z_p|^2 - 1}{|z_p|^2 - R^2} - 1 \right) k$$

or

$$n(|z_p|^2 - R^2) > k(|z_p|^2 - 1),$$

or

$$|z_p|^2 \left(1 - \frac{k}{n}\right) + \frac{k}{n} > R^2. \quad (16)$$

To prove (16), rewrite (14) in the form

$$R^2 \leq \left(1 - \frac{k}{n}\right)^2 |z_p|^2 - 2\frac{k}{n} \left(1 - \frac{k}{n}\right) |z_p| + \frac{k^2}{n^2}. \quad (17)$$

Note that the left-hand side of (16) is greater than the right-hand side of (17):

$$|z_p|^2 \left(1 - \frac{k}{n}\right) + \frac{k}{n} > \left(1 - \frac{k}{n}\right)^2 |z_p|^2 - 2\frac{k}{n} \left(1 - \frac{k}{n}\right) |z_p| + \frac{k^2}{n^2},$$

because the last inequality is equivalent to

$$\frac{k}{n} \left(1 - \frac{k}{n}\right) |z_p|^2 + 2\frac{k}{n} \left(1 - \frac{k}{n}\right) |z_p| + \frac{k}{n} \left(1 - \frac{k}{n}\right) > 0$$

or

$$(|z_p| + 1)^2 > 0.$$

Therefore, (16) is true, and we have proved that $c_p > 0$.

It remains to show that $c_p \geq r_p$, i. e.,

$$\frac{n - k}{n} \frac{R^2}{R^2 - 1} - \frac{R^2}{n} \sum_{j=1}^p \frac{k_j}{|z_j|^2 - R^2} \geq \frac{n - k}{n} \frac{R}{R^2 - 1} + \frac{R}{n} \sum_{j=1}^p \frac{k_j |z_j|}{|z_j|^2 - R^2}.$$

Rewrite the last inequality in the equivalent form:

$$\frac{n - k}{R + 1} \geq \sum_{j=1}^p \frac{k_j}{|z_j| - R}. \quad (18)$$

Inequality (18) follows from the inequality

$$n - k \geq (R + 1) \sum_{j=1}^p \frac{k_j}{|z_p| - R} = \frac{k(R + 1)}{|z_p| - R}. \quad (19)$$

Let us check the validity of (19) for $R \in \left(1, \left(1 - \frac{k}{n}\right)|z_p| - \frac{k}{n}\right)$. Since the function $\frac{R + 1}{|z_p| - R}$ increases on R , it is sufficient to check (19) for $R = \left(1 - \frac{k}{n}\right)|z_p| - \frac{k}{n}$. For such R , inequality (19) is

$$(n - k) \left(|z_p| - \left(1 - \frac{k}{n}\right)|z_p| + \frac{k}{n} \right) \geq k \left(\left(1 - \frac{k}{n}\right)|z_p| - \frac{k}{n} + 1 \right). \quad (20)$$

Simple calculations show that (20) is an equality. Hence, $c_p \geq r_p$. \square

Remark. Note that Lemma 2 works only if $\left(1 - \frac{k}{n}\right)|z_p| - \frac{k}{n} > 1$, i. e.,

$$|z_p| > \frac{n + k}{n - k}.$$

Now we have everything to obtain the following refinement of the Bernstein inequality (2):

Theorem 2. Let $0 < r < 1$, f and F be polynomials, such that

- 1) $\deg f \leq \deg F = n$;
- 2) all the zeros of F belong to $\overline{\mathbb{D}}$; moreover, let z_1, \dots, z_q be all the zeros of F of orders k_1, \dots, k_q , $1 \leq k_1 + \dots + k_q \leq n$, such that $r < |z_q| < \dots < |z_1| \leq 1$;
- 3) $|f(z)| \leq |F(z)|$ for $|z| \geq r$.

Then

$$|f'(z)| \leq |F'(z)|$$

for all z , $|z| = R$, where

$$R \in [|z_1|, \infty) \cup \left(\bigcup_{m=1}^{q-1} (|z_{m+1}|, |z_m|) \setminus I_m \right) \cup ((r, |z_q|) \setminus I_q). \quad (21)$$

Here

$$I_1 = \left(\left(1 - \frac{k_1}{n}\right)|z_1| - \frac{k_1}{n}|z_2|, \left(1 - \frac{k_1}{n}\right)|z_1| + \frac{k_1}{n}|z_2| \right),$$

$$I_m = \left(\left(1 - \frac{k_1 + \dots + k_m}{n} \right) |z_m| - \frac{k_1 + \dots + k_m}{n} |z_{m+1}|, |z_m| \right),$$

$$m = 2, \dots, q-1,$$

$$I_q = \left(\left(1 - \frac{k_1 + \dots + k_q}{n} \right) |z_q| - \frac{k_1 + \dots + k_q}{n} r, |z_q| \right).$$

Proof. 1) Firstly, consider the case $R \in [|z_1|, +\infty)$. Put $f_1(w) = f(|z_1|w)$, $F_1(w) = F(|z_1|w)$. For these polynomials, all assumptions of Theorem E are satisfied. Therefore, by Theorem E,

$$|S_\alpha[f_1](w)| \leq |S_\alpha[F_1](w)|, \quad |w| = R_1 \geq 1,$$

where $\alpha \in \Omega_{R_1}$. This is equivalent to

$$|w|z_1|f'(|z_1|w) - \alpha n f(|z_1|w)| \leq |w|z_1|F'(|z_1|w) - \alpha n F(|z_1|w)|,$$

$|w| = R_1 \geq 1$, $\alpha \in \Omega_{R_1}$. Denoting $z = w|z_1|$, $R = R_1|z_1|$, we have

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)|, \quad |z| = R \geq |z_1|,$$

for $\alpha \in \Omega_{\frac{R}{|z_1|}}$. Since $0 \in \Omega_{\frac{R}{|z_1|}}$, we obtain

$$|f'(z)| \leq |F'(z)|, \quad |z| = R \geq |z_1|.$$

2) Now let $R \in (|z_2|, |z_1|)$. The polynomials $f_2(w) = f(|z_2|w)$, $F_2(w) = F(|z_2|w)$ satisfy the conditions of Lemma 1, the polynomial F_2 has the unique zero $w = \frac{z_1}{|z_2|}$ outside \mathbb{D} . The order of this zero equals k_1 . By Lemma 1, $D(R_2, 1, n)$ contains 0 if

$$R_2 \in \left(1, \left| \frac{z_1}{z_2} \right| \right) \setminus \left(\left(1 - \frac{k_1}{n} \right) \left| \frac{z_1}{z_2} \right| - \frac{k_1}{n}, \left(1 - \frac{k_1}{n} \right) \left| \frac{z_1}{z_2} \right| + \frac{k_1}{n} \right).$$

Therefore,

$$|f'_2(w)| \leq |F'_2(w)|, \quad |w| = R_2, \tag{22}$$

for these R_2 . Denoting $z = w|z_2|$, $R = R_2|z_2|$, rewrite (22) in the form $|f'(z)| \leq |F'(z)|$. Then the last inequality takes place for $|z| = R \in (|z_2|, |z_1|) \setminus I_1$.

3) If $R \in (|z_{m+1}|, |z_m|)$, $m = 2, \dots, q-1$, then we consider the following polynomials: $f_{m+1}(w) = f(|z_{m+1}|w)$ and $F_{m+1}(w) = F(|z_{m+1}|w)$. They satisfy the conditions of Lemma 2 with $p = m$. The polynomial F_m

has m zeros $\frac{z_1}{|z_{m+1}|}, \dots, \frac{z_m}{|z_{m+1}|}$ of orders k_1, \dots, k_m outside $\overline{\mathbb{D}}$. Applying Lemma 2, we obtain

$$|f'_{m+1}(w)| \leq |F'_{m+1}(w)|, \quad |w| = R_{m+1}, \quad (23)$$

for

$$R_m \in \left(1, \left|\frac{z_m}{z_{m+1}}\right|\right) \setminus \left(\left(1 - \frac{k_1 + \dots + k_m}{n}\right) \left|\frac{z_m}{z_{m+1}}\right| - \frac{k_1 + \dots + k_m}{n}; +\infty\right).$$

Denote $z = w|z_{m+1}|$, $R = R_{m+1}|z_{m+1}|$; from (23) we obtain $|f'(z)| \leq |F'(z)|$ for $|z| = R$, where $R \in (|z_{m+1}|, |z_m|) \setminus I_m$.

4) For $R \in (r, |z_q|)$ we repeat the reasonings from the previous step for the polynomials $f_r(w) = f(rw)$ and $F_r(w) = F(rw)$, and obtain the Bernstein inequality for $R \in (r, |z_q|) \setminus I_q$. \square

Remark. If polynomials f and F satisfy the conditions of Theorem 2 and the inequality

$$\min \left\{ \left| \frac{z_1}{z_2} \right|, \left| \frac{z_2}{z_3} \right|, \dots, \left| \frac{z_{q-1}}{z_q} \right|, \left| \frac{z_q}{r} \right| \right\} > \frac{n + k_1 + \dots + k_q}{n - k_1 - \dots - k_q}$$

takes place, then (21) implies that inequality (2) is true not only in the complement to \mathbb{D} , but also in the following rings that are in \mathbb{D} :

$$\Gamma_1 = \left\{ z \in \mathbb{C} : \left(1 - \frac{k_1}{n}\right) |z_1| + \frac{k_1}{n} |z_2| \leq |z| < 1 \right\},$$

$$\Gamma_2 = \left\{ z \in \mathbb{C} : |z_2| < |z| \leq \left(1 - \frac{k_1}{n}\right) |z_1| - \frac{k_1}{n} |z_2| \right\},$$

...

$$\Gamma_q = \left\{ z \in \mathbb{C} : |z_q| < |z| \leq \left(1 - \frac{k_1 + \dots + k_{q-1}}{n}\right) |z_{q-1}| - \frac{k_1 + \dots + k_{q-1}}{n} |z_q| \right\},$$

$$\Gamma_{q+1} = \left\{ z \in \mathbb{C} : r < |z| \leq \left(1 - \frac{k_1 + \dots + k_q}{n}\right) |z_q| - \frac{k_1 + \dots + k_q}{n} r \right\}.$$

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