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ON STRONGLY ALMOST CONVERGENCE OF DOUBLE SEQUENCES VIA COMPLEX UNCERTAIN VARIABLE

Abstract. The aim of this article is to introduce the characterization of strongly almost convergence in complex uncertain double sequences. We investigate the concept of strongly almost convergence almost surely, convergence in measure, convergence in mean, convergence in distribution, and convergence uniformly almost surely of complex uncertain double sequences. Finally, we present the interrelationships among those newly introduced notions.

Key words: *uncertain variables, uncertainty space, complex uncertain double sequence, strongly almost convergence*

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1. Introduction. Convergence of sequences plays a significant role in the fundamental theory of mathematical analysis. The notion of strongly almost convergence of real sequences was introduced by Maddox [14] in 1979. It is also reported by Lorentz [13] and later it was studied from the sequence-space point of view via summability by Mursaleen [17], King [8], Savas [21], and many others. The idea of the double sequence, which is a double infinite array, and its convergence was introduced by Pringsheim [20]. Some primary works on double-sequence spaces can be found in Bromwich [1] and thereafter it was investigated by Hardy [7], Moricz and Rhoades [16], Tripathy [22], and many more.

Uncertainty theory was initiated by Liu in 2007 and the latest 5th edition was published in 2016 [12]. Again, he extended this work on uncertain measure, which is a set function that follows normality, monotonicity, self-duality, countable sub-additivity, and product axioms. Later the notion of uncertain variables is investigated to represent the uncertain quantity and to describe the uncertainty distribution. Nowadays, uncertain theory has become a branch of mathematics and many developments are made within the uncertain theory based on various fields, like uncertain set [12], uncertain inference [10], uncertain logic [11], uncertain entailment [9], uncertain calculus [12], uncertain differential equation [2], uncertain statistics [4], and many more. Undoubtedly, these concepts play a vital role in real-life problems and applications.

The uncertainty theory appears not only in real-life quantities but also in complex uncertain quantities. Peng [19] introduced in 2012 the idea of complex uncertain variables, which are measurable functions from an uncertainty space to the set of complex numbers, and proposed the notion of the complex uncertain expected value, complex uncertain measure, and complex uncertain distribution.

Liu [12] initiated and reported four types of convergence concepts on sequences: viz. convergence in almost surely, convergence in measure, convergence in mean, convergence in distribution. Thereafter, You [24] expanded this work by introducing the notion of convergence in uniformly almost surely and established interrelationships between these convergences and other existing ones. Moreover, Chen et al. [3] explored the idea of convergence in the complex uncertain sequences using the complex uncertain variable. Most recent research works on complex uncertain sequences are also found in [5], [6], [18], [23], and many more.

In this paper, our primary focus is to study the concepts of strongly almost-convergence of complex uncertain double sequences and other types of convergence concepts, like convergence in almost surely, convergence in measure, convergence in mean, convergence in distribution, and convergence uniformly almost surely for a given complex uncertain double sequence. Finally, we discuss the interrelationships among these newly defined notions.

2. Preliminaries. In this section, some basic concepts on the double sequence and the uncertainty theory used throughout this paper are recalled.

Definition 1. *[22] A double sequence can be written as a double infinite array $< a_{mn} >$ for all $m, n \in \mathbb{N}$. A double sequence $< a_{mn} >$ is said to converge in Pringsheim's sense [20] if $\lim_{m,n\to\infty} a_{mn} = L$ exists, where m, n tend to ∞ independently of each other.*

A double sequence $< a_{mn} >$ is said to be bounded if there exists $M > 0$ such that

$$\sup_{m,n} |a_{m,n}| < M.$$

A double sequence $< a_{mn} >$ *is said to be regularly convergent (Hardy [7]), if it converges in Pringsheim's sense and the following limits hold:*

$$\lim_{n\to\infty} a_{mn} = L_m, \text{ exists for } m \in \mathbb{N} \text{ and } \lim_{m\to\infty} a_{mn} = L_n, \text{ exists for } n \in \mathbb{N}.$$

A double sequence is said to be Cesàro summable [15] to L *if,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = L.$$

A double sequence is said to be strongly almost convergent to a number L *if*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |a_{mn} - L| = 0, \text{ uniformly in } m,n.$$

Definition 2. *[12] Let* $\mathscr{L}$ *be a* σ*-algebra on a non-empty set* Γ*. A set function* $\mathscr{M}$ *on* Γ *is called an uncertain measure if it satisfies the following axioms:*

Axiom 1 (Normality Axiom) $\mathscr{M}\{\Gamma\}=1$;

Axiom 2 (Duality Axiom): $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\}=1$ *for any* $\Lambda \in \mathscr{L}$;

Axiom 3 (Sub-additivity Axiom): For every countable sequence of events $\{\Lambda_j\} \in \mathscr{L}$*, we have*

$$\mathscr{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leqslant \sum_{j=1}^{\infty} \mathscr{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathscr{L}, \mathscr{M})$ *is called an uncertainty space, and each element* Λ *in* $\mathscr{L}$ *is called an uncertain event. To obtain an uncertain measure of compound events, a product uncertain measure is defined as follows:*

Axiom 4 (Product Axiom): Let $(\Gamma_k, \mathscr{L}_k, \mathscr{M}_k)$ *be uncertainty spaces for* $k = 1, 2, 3 \ldots$ *The product uncertain measure* $\mathscr{M}$ *is an uncertain measure satisfying*

$$\mathscr{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathscr{M}\{\Lambda_j\},$$

where Λ_j *are arbitrarily chosen events from* Γ_j *for* $j = 1, 2, 3, \ldots,$ *respectively.*

Definition 3. *[19] A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of complex numbers, i. e., for any Borel set B of complex numbers, the set $\{\zeta \in B\}=\{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.*

Definition 4. *[12] The expected value operator of a complex uncertain variable ζ is defined by*

$$E[\zeta] = \int_{0}^{+\infty} \mathscr{M}\{\zeta \geqslant r\}dr - \int_{-\infty}^{0} \mathscr{M}\{\zeta \leqslant r\}dr,$$

provided that at least one of the two integrals is finite.

The complex uncertain distribution $\varphi(c)$ of a complex uncertain variable ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by

$$\varphi(c) = \mathscr{M}\big\{\operatorname{Re}(\zeta) \leqslant \operatorname{Re}(c), \operatorname{Im}(\zeta) \leqslant \operatorname{Im}(c)\big\},$$

for any complex number c.

Theorem 1. *[19] A function $\varphi(c) \to [0, 1]$ is a complex uncertainty distribution if and only if it is increasing with respect to the real part $\operatorname{Re}(c)$ and the imaginary part $\operatorname{Im}(c)$ and is such that*

$$\lim_{x\to\infty} \varphi(x + ib) \neq 1, \ \lim_{y\to\infty} \varphi(a + iy) \neq 1, \text{ for any } a, b \in \mathbb{R},$$
$$\lim_{x\to\infty, y\to\infty} \varphi(x + iy) \neq 0, \ \text{ where } i = \sqrt{-1} \text{ is the imaginary unit.}$$

Chen et al. [3] introduced the concepts of convergence for complex uncertain sequences as follows:

Definition 5. *[3] The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely to ζ if there exists an event Λ with $\mathscr{M}\{\Lambda\}=1$, such that*

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0,$$

for every $\gamma \in \Lambda$. In that case, we write $\zeta_n \to \zeta$ almost surely.

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent to ζ in measure if for any $\varepsilon \geqslant 0$

$$\lim_{n\to\infty} \mathscr{M}\big\{\|\zeta_n - \zeta\| \geqslant \varepsilon\big\} = 0.$$

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent to ζ in mean if

$$\lim_{n\to\infty} E\big[\|\zeta_n - \zeta\|\big] = 0.$$

Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be complex uncertainty distributions of complex uncertain variables $\zeta_1, \zeta_2, \zeta_3 \ldots$, respectively. Then the complex uncertain sequence $\{\zeta_n\}$ is convergent to ζ in distribution if

$$\lim_{n\to\infty} \varphi_n(c) = \varphi(c) \text{ for all } c \in \mathbb{C} \text{ where } \varphi(c) \text{ is continuous.}$$

The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to ζ if there exists a sequence of events $\{E_k\}$ with $\mathscr{M}\{E_k\} \to 0$, such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - E_k$ for any fixed $k \in \mathbb{N}$.

3. Main findings. In this section, we introduce the concept of strongly almost convergence of complex uncertain double sequence in different aspects. Also, we study some important results and establish the interrelationships among these notions.

Definition 6. *The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variables is said to be strongly almost convergent to ζ with respect to almost surely convergence, if there exists an event Λ with $\mathscr{M}\{\Lambda\} = 1$, such that for every $\gamma \in \Lambda$:*

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \|\zeta_{m+i,\ n+j}(\gamma) - \zeta(\gamma)\| = 0, \text{ uniformly in } m, n.$$

The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variables is said to be strongly almost convergent to ζ in measure if for every $\varepsilon > 0$, such that

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\big\{\|\zeta_{m+i,n+j} - \zeta\| \geqslant \varepsilon\big\} = 0, \text{ uniformly in } m, n.$$

The double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is said to be strongly almost convergent to ζ in mean, if

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} E\big[\|\zeta_{m+i,n+j} - \zeta\|\big] = 0, \text{ uniformly in } m, n.$$

Let φ and $\varphi_{m,n}$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_{m,n}$ respectively. Then, for any $n = 1, 2, 3, \ldots$, *the complex uncertain double sequence* $\{\zeta_{m,n}\}$ *is said to be strongly almost convergent to* ζ *in distribution, if*

$$\lim_{\substack{p\to\infty\\ q\to\infty}} \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \|\varphi_{m+i,\ n+j}(c) - \varphi(c)\| = 0,$$

uniformly in m, n *and* $\forall\ c$ *at which* $\varphi(c)$ *is continuous.*

The double sequence $\{\zeta_{m,n}\}$ *of complex uncertain variables is said to be strongly almost convergent to* ζ *with respect to uniformly almost surely convergence if there exist* $\{E_k\}$ *with* $\mathscr{M}\{E_k\} \to 0$, *such that* $\{\zeta_{m,n}\}$ *is strongly almost convergent uniformly in* $\Gamma - E_k$ *for each fixed* k.

Moving to the main results, we first consider strongly almost convergence in mean and that of in measure for the complex uncertain double sequence.

Theorem 2. *If the double sequence* $\{\zeta_{m,n}\}$ *of complex uncertain variables is strongly almost convergent to* ζ *in mean, then* $\{\zeta_{m,n}\}$ *is strongly almost convergent to* ζ *in measure, but the converse may not be true in general.*

Proof. Let the double sequence $\{\zeta_{m,n}\}$ of complex uncertain variables be strongly almost convergent in mean. By the Markov inequality, for every given $\varepsilon > 0$ and uniformly in m, n, we have:

$$\mathscr{M}\big\{\|\zeta_{m,n} - \zeta\| \geqslant \varepsilon\big\} \leqslant \frac{E\big[\|\zeta_{m,n} - \zeta\|\big]}{\varepsilon}, \quad \text{or,}$$

$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \mathscr{M}\big\{\|\zeta_{m+i,\ n+j} - \zeta\| \geqslant \varepsilon\big\} \leqslant \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \frac{E\big[\|\zeta_{m+i,\ n+j} - \zeta\|\big]}{\varepsilon} \to 0,$$

as $p, q \to \infty$.

Thus, the double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ in measure. Therefore, strongly almost convergence in measure follows from the almost convergence in mean. □

To show that the converse is not true in general, we provide the following counter-example.

Example 3. Consider the uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$, $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$.

Define the uncertain measure

$$\mathscr{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_{m+n}\in\Lambda} \dfrac{1}{2(m+n)^2+1}, & \text{if } \sup\limits_{\gamma_{m+n}\in\Lambda} \dfrac{1}{2(m+n)^2+1} < \dfrac{1}{2}; \\ 1 - \sup\limits_{\gamma_{m+n}\in\Lambda^c} \dfrac{1}{2(m+n)^2+1}, & \text{if } \sup\limits_{\gamma_{m+n}\in\Lambda^c} \dfrac{1}{2(m+n)^2+1} < \dfrac{1}{2}; \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Consider the complex uncertain variables $\zeta_{m,n}$ given by

$$\zeta_{m,n}(\gamma) = \begin{cases} \{2(m+n)^2+1\}i, & \text{if } \gamma = \gamma_{m+n}; \\ 0, & \text{otherwise,} \end{cases}$$

$\forall\ m,n \in \mathbb{N}$ and $\zeta(\gamma) \equiv 0$, $\forall \gamma \in \Gamma$. Again, for any $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$ and $m,n \geqslant n_0$ such that $\frac{1}{2(m+n)^2+1} < \frac{1}{2}$. Then,

$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \mathscr{M}\left\{\gamma\colon \|\zeta_{m+i,\,n+j}(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\right\} = \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \frac{1}{2(i+j)^2+1} \to 0,$$

as $p,q \to \infty$ uniformly in m,n, since the Cesàro mean is a regular method and the original sequence converges to 0. Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ in measure.

Now, the uncertain distribution function $\varphi_{m,n}$ for the complex uncertain variable $\zeta_{m,n}$ is

$$\varphi_{m,n}(x) = \begin{cases} 0, & \text{if } x < 0\ ; \\ 1 - \dfrac{1}{2(m+n)^2+1}, & \text{if } 0 \leqslant x < 2(m+n)^2+1; \\ 1, & \text{if } x \geqslant 2(m+n)^2+1. \end{cases}$$

Then, we have

$$E\big[\|\zeta_{m,n} - \zeta\|\big] = \int\limits_0^{2(m+n)^2+1} \Big[1 - (1 - \frac{1}{2(m+n)^2+1})\Big] dx +$$

$$+ \int\limits_{2(m+n)^2+1}^{\infty} (1-1)dx - \int\limits_{-\infty}^{0} 0\ dx = 1.$$

That is, $\int\limits_{0}^{2(m+n)^2+1} \frac{1}{2(m+n)^2+1} = 1$, uniformly in m, n.

Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent to ζ in mean.

Theorem 3. *Let $\{\zeta_{m,n}\}$ be a double sequence of complex uncertain variable with the real part $\{\xi_{m,n}\}$ and the imaginary part $\{\eta_{m,n}\}$, respectively, for m, $n = 1, 2, 3, \ldots$ If the uncertain sequences $\{\xi_{m,n}\}$ and $\{\eta_{m,n}\}$ are strongly almost convergent in measure to ξ and η, respectively, if and only if the complex uncertain double sequence $\{\zeta_{m,n}\}$ is also strongly almost convergent in measure to $\zeta = \xi + i\eta$.*

Proof. Let the double sequence $\{\xi_{m,n}\}$ and $\{\eta_{m,n}\}$ be strongly almost convergent in measure to ξ and η respectively. Then, for every $\varepsilon > 0$,

$$\lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\Big\{|\xi_{m+i,\,n+j} - \xi| \geqslant \frac{\varepsilon}{\sqrt{2}}\Big\} = 0, \text{ uniformly in } m, n,$$

$$\lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\Big\{|\eta_{m+i,\,n+j} - \eta| \geqslant \frac{\varepsilon}{\sqrt{2}}\Big\} = 0, \text{ uniformly in } m, n.$$

Here, $\|\zeta_{m+i,\,n+j} - \zeta\| = \sqrt{|\xi_{m+i,\,n+j} - \xi|^2 + |\eta_{m+i,\,n+j} - \eta|^2}$.

We have:

$$\{\|\zeta_{m+i,\,n+j} - \zeta\| \geqslant \varepsilon\} \subset \Big\{|\xi_{m+i,\,n+j} - \xi| \geqslant \frac{\varepsilon}{\sqrt{2}}\Big\} \cup \Big\{|\eta_{m+i,\,n+j} - \eta| \geqslant \frac{\varepsilon}{\sqrt{2}}\Big\}.$$

Now, using the sub-additivity axiom of the uncertain measure, we get:

$$\lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\left\{\|\zeta_{m+i,\,n+j} - \zeta\| \geqslant \varepsilon\right\} \leqslant$$

$$\leqslant \lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\{|\xi_{m+i,\,n+j} - \xi| \geqslant \frac{\varepsilon}{\sqrt{2}}\}+$$

$$+ \lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\{|\eta_{m+i,\,n+j} - \eta| \geqslant \frac{\varepsilon}{\sqrt{2}}\}.$$

So, $\lim\limits_{p,\,q\to\infty} \frac{1}{pq} \sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} \mathscr{M}\left\{\|\zeta_{m+i,\;n+j} - \zeta\| \geqslant \varepsilon\right\} = 0$, uniformly in m, n.

Consequently, the double sequence $\{\zeta_{m,n}\}$ of complex uncertain variable is strongly almost convergent in measure to ζ.

Conversely, let the complex uncertain sequence $\{\zeta_{m,n}\}$ be strongly almost convergent to ζ in measure. Then we have for any $\delta > 0$:

$$\lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{\|\zeta_{m+i,\,n+j}-\zeta\|\geqslant\delta\big\}=0,\quad \text{uniformly in } m,n.$$

$$\Rightarrow \lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{|(\xi_{m+i,\,n+j}+i\eta_{m+i,\ n+j})-(\xi+i\eta)|\geqslant\delta\big\}=0$$

$$\Rightarrow \lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{|(\xi_{m+i,\,n+j}-\xi)+i(\eta_{m+i,\,n+j}-\eta)|\geqslant\delta\big\}=0.$$

Then, there exists a positive number δ' with $0<\delta^{'}<\frac{\delta}{2}$, such that

$$\lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{|\xi_{m+i,\,n+j}-\xi|\geqslant\delta^{'}\big\}=0,\quad \text{uniformly in } m,n,$$

$$\text{and}\quad \lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{|\eta_{m+i,\,n+j}-\eta|\geqslant\delta^{'}\big\}=0,\ \text{uniformly in } m,n.$$

Hence, the real part $\{\xi_{m,n}\}$ and the imaginary part $\{\eta_{m,n}\}$ of the complex uncertain sequence $\{\zeta_{m,n}\}$ are strongly almost convergent in measure to ξ and η, respectively. □

Theorem 4. *If a complex uncertain double sequence $\{\zeta_{m,n}\}$ with the real part $\{\xi_{m,n}\}$ and the imaginary part $\{\eta_{m,n}\}$ is strongly almost convergent in measure to ξ and η, respectively, then the double sequence convergent in distribution.*

Proof. Let $c = a + ib$ be a given point of continuity of the complex uncertainty distribution φ for $\zeta = \xi + i\eta$. For any $\alpha > a$, $\beta > b$, we have:

$$\begin{aligned}
&\big\{\xi_{m,n}\leqslant a,\eta_{m,n}\leqslant b\big\}=\\
&=\big\{\xi_{m,n}\leqslant a,\eta_{m,n}\leqslant b,\xi\leqslant\alpha,\eta\leqslant\beta\big\}\cup\big\{\xi_{m,n}\leqslant a,\eta_{m,n}\leqslant b,\xi>\alpha,\eta>\beta\big\}\cup\\
&\cup\big\{\xi_{m,n}\leqslant a,\eta_{m,n}\leqslant b,\xi\leqslant\alpha,\eta>\beta\big\}\cup\big\{\xi_{m,n}\leqslant a,\eta_{m,n}\leqslant b,\xi>\alpha,\eta\leqslant\beta\big\}\subset\\
&\qquad\subset\big\{\xi\leqslant\alpha,\eta\leqslant\beta\big\}\cup\big\{|\xi_{m,n}-\xi|\geqslant\alpha-a\big\}\cup\big\{|\eta_{m,n}-\eta|\geqslant\beta-b\big\}.
\end{aligned}$$

Also, it follows from the subadditivity axiom that

$$\begin{aligned}\varphi_{m,n}(c) = \varphi_{m,n}(a+ib) \leqslant \varphi(\alpha+i\beta)+ \\ + \mathscr{M}\{|\xi_{m+i,n+j}-\xi| \geqslant \alpha - a\} + +\mathscr{M}\{|\eta_{m+i,n+j}-\eta| \geqslant \beta - b\}.\end{aligned}$$

Since the real and imaginary parts of the complex uncertain double sequence are strongly almost convergent in measure, we have, for $\varepsilon > 0$:

$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\{|\xi_{m+i,n+j}-\xi| \geqslant \alpha - a\} < \frac{\varepsilon}{2},$$
$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\{|\eta_{m+i,n+j}-\eta| \geqslant \beta - b\} < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\lim_{p,\,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\varphi_{m,n}(c) \leqslant \varphi(\alpha+i\beta)+ \\ + \lim_{p,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\{|\xi_{m+i,\ n+j}-\xi| \geqslant \alpha - a\}+ \\ + \lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\mathscr{M}\{|\eta_{m+i,\ n+j}-\eta| \geqslant \beta - b\} < \\ < \varphi(\alpha+i\beta)+\varepsilon/2+\varepsilon/2 = \varphi(\alpha+i\beta)+\varepsilon.\end{aligned}$$

Suppose, $\alpha + i\beta \to a + ib$. We obtain:

$$\lim_{p,\,q\to\infty}\sup\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\varphi_{m+i,\ n+j}(c)\} \leqslant \varphi(\alpha+i\beta) = \varphi(c), \text{ uniformly in } m,n. \tag{1}$$

Again, for $x < a$, $y < b$, we have:

$$\begin{aligned}&\big\{\xi_{m,n} \leqslant x, \eta_{m,n} \leqslant y\big\} = \\ &= \big\{\xi_{m,n} \leqslant a, \eta_{m,n} \leqslant b, \xi \leqslant x, \eta \leqslant y\big\} \cup \big\{\xi_{m,n} \leqslant a, \eta_{m,n} > b, \xi \leqslant x, \eta \leqslant y\big\} \cup \\ &\cup \big\{\xi_{m,n} \leqslant a, \eta_{m,n} \leqslant b, \xi \leqslant \alpha, \eta > \beta\big\} \cup \big\{\xi_{m,n} \leqslant a, \eta_{m,n} \leqslant b, \xi > \alpha, \eta \leqslant \beta\big\} \\ &\subset \big\{\xi_{m,n} \leqslant a, \eta_{m,n} \leqslant b\big\} \cup \big\{|\xi_{m,n}-\xi| \geqslant a - x\big\} \cup \big\{|\eta_{m,n}-\eta| \geqslant b - y\big\}.\end{aligned}$$

This implies:

$$\varphi(x+iy) \leqslant \varphi_{m,n}(a+ib) + \mathscr{M}\{|\xi_{m,n}-\xi| \geqslant a-x\} + \mathscr{M}\{|\eta_{m,n}-\eta| \geqslant b-y\}.$$

Thus, like in the previous step, we have:

$$\varphi(x+iy) \leqslant \lim_{p,\,q\to\infty} \inf \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\varphi_{m+i,n+j}(a+ib), \text{ uniformly in } m,n.$$

Then, for any $x < a$, $y < b$, we obtain $x + iy \to a + ib$; so,

$$\varphi(c) \leqslant \lim_{p,\,q\to\infty} \inf \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\varphi_{m+i,n+j}(c), \quad \text{for uniformly in } m,n. \tag{2}$$

From equations (1) and (2) we see that $\varphi_{m,n}(c)$ is strongly almost converges in distribution to $\varphi(c)$ as $m, n \to \infty$. That is, the real and imaginary parts are strongly almost convergent in measure and, consequently, strongly almost convergent in distribution. □

For the converse part, we consider the following example:

Example 4. Consider the uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, having $\mathscr{M}\{\gamma_1\} = 0.7$, $\mathscr{M}\{\gamma_2\} = 0.2$, and $\mathscr{M}\{\gamma_3\} = 0.3$. Define the sequence of complex uncertain variable by

$$\zeta_{m,n}(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1; \\ -i, & \text{if } \gamma = \gamma_2; \\ 2i, & \text{if } \gamma = \gamma_3. \end{cases}$$

We also have $\zeta = -\zeta_{m,n}$ for $m, n \in \mathbb{N}$. Then $\zeta_{m,n}$ and ζ have the same distribution as

$$\varphi_{m,n}(c) = \varphi_{m,n}(a+ib) = \begin{cases} 0, & \text{if } a < 0,\ b < \infty; \\ 0, & \text{if } a \geqslant 0,\ b < -1; \\ 0.2, & \text{if } a \geqslant 0,\ -1 \leqslant b < 1; \\ 0.7, & \text{if } a \geqslant 0,\ 1 \leqslant b < 2; \\ 1, & \text{if } a \geqslant 0, 1,\ b \geqslant 2. \end{cases}$$

Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ in distribution.

Now, for any $\varepsilon > 0$ and for p, $q \to \infty$, we have

$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{\|\zeta_{m+i,n+j}-\zeta\| \geqslant \varepsilon\big\} =$$

$$= \frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\mathscr{M}\big\{\gamma\colon \|\zeta_{m+i,n+j}(\gamma)-\zeta(\gamma)\| \geqslant \varepsilon\big\} = 1$$

uniformly in m, n. Thus, the double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent to ζ in measure.

Remark 1. *Strongly almost convergence with respect to almost surely convergence does not imply strongly almost convergence in mean, strongly almost convergence in measure, and strongly almost convergence in distribution. This result is illustrated in the following example.*

Example 5. Let the uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ be $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ with $\mathscr{M}\{\Lambda\} = \sum_{\gamma\in\Lambda} \frac{1}{3^{(m+n)}}$. The complex uncertain variables are defined by

$$\zeta_{m,n}(\gamma) = \begin{cases} i3^{(m+n)}, & \text{if } \gamma = \gamma_{m+n};\\ 0, & \text{otherwise,}\end{cases}$$

for $m, n \in \mathbb{N}$ and $\zeta(\gamma) \equiv 0$. Then the double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent with respect to almost surely convergence to ζ. However, for $m, n \in \mathbb{N}$, the uncertainty distribution of $\zeta_{m,n}$ is

$$\varphi_{m,n}(x) = \begin{cases} 0, & \text{if } x < 0;\\ 1-\frac{1}{3^{(m+n)}}, & \text{if } 0 \leqslant x < 3^{(m+n)};\\ 1, & \text{if } x \geqslant 3^{(m+n)}.\end{cases}$$

Then, we have

$$E\big[\|\zeta_{m+i,n+j}-\zeta\|\big] = \int_0^{3(m+n)} \big[1-(1-\frac{1}{3^{(m+n)}})\big]dx + \int_{3^{(m+n)}}^{\infty}(1-1)dx - \int_{-\infty}^{0} 0\,dx = 1,$$

$$\text{i.e., } \lim_{p,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}E\big[\|\zeta_{m+i,n+j}-\zeta\|\big] = 1, \quad \text{uniformly in } m, n.$$

Consequently, the double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent to ζ in mean.

Example 6. Let the uncertainty space$(\Gamma, \mathscr{L}, \mathscr{M})$ be $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with $\mathscr{M}\{\gamma_1\} = 0.6$, $\mathscr{M}\{\gamma_2\} = 0.4$, $\mathscr{M}\{\gamma_3\} = 0.4$, $\mathscr{M}\{\gamma_4\} = 0.4$. Define the sequence of complex uncertain variable by

$$\zeta_{m,n}(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1; \\ 2i, & \text{if } \gamma = \gamma_2; \\ 3i, & \text{if } \gamma = \gamma_3; \\ 4i, & \text{if } \gamma = \gamma_4; \\ 0, & \text{otherwise,} \end{cases}$$

for $m, n \in \mathbb{N}$ and $\zeta \equiv 0$. Then the double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ with respect to almost surely convergence.

However, we have, for a given $\varepsilon > 0$:

$$\mathscr{M}\Big\{\|\zeta_{m+i,\ n+j} - \zeta\| \geqslant \varepsilon\Big\} = \mathscr{M}\Big\{\gamma\colon \|\zeta_{m+i,\ n+j}(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\Big\} = 1, \text{i. e.,}$$

$$\lim_{p,\,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\Big\{\gamma\colon \|\zeta_{m+i,\,n+j}(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\Big\} = 1, \text{ uniformly in } m, n.$$

Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent in measure. Let the complex uncertainty distribution of $\{\zeta_{m,n}\}$ be defined by

$$\varphi_{m,n}(c) = \varphi_{m,n}(a + ib) = \begin{cases} 0, & \text{if } a < 0,\ b < \infty; \\ 0, & \text{if } a \geqslant 0,\ b < 1; \\ 0.6, & \text{if } a \geqslant 0,\ 1 \leqslant b < 2; \\ 0.6, & \text{if } a \geqslant 0,\ 2 \leqslant b < 3; \\ 0.6, & \text{if } a \geqslant 0,\ 3 \leqslant b < 4; \\ 1, & \text{if } a \geqslant 0,\ b \geqslant 4. \end{cases}$$

The complex uncertain distribution for ζ be defined by

$$\varphi(c) = \begin{cases} 0, & \text{if } a < 0,\ b < \infty; \\ 0, & \text{if } a \geqslant 0,\ b < 0; \\ 1, & \text{if } a \geqslant 0,\ b \geqslant 0. \end{cases}$$

Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent in distribution.

Remark 2. *The strongly almost convergence in measure and strongly almost convergence in mean do not imply strongly almost convergence with respect to almost surely convergence. This result can be established from the following example:*

Example 7. Consider an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to be $[0,1]$. Define a complex uncertain variable by

$$\zeta_{m,n}(\gamma) = \begin{cases} i, & \text{if } \dfrac{t}{2^{k_1+k_2}} \leqslant \gamma \leqslant \dfrac{(1+t)}{2^{k_1+k_2}}; \\ 0, & \text{otherwise.} \end{cases}$$

Then for all $m = 2^{k_1} + t, n = 2^{k_2} + t \in \mathbb{N}$ and $\zeta \equiv 0$. For a given $\varepsilon > 0$, we have

$$\mathscr{M}\big\{\|\zeta_{m+i,n+j} - \zeta\| \geqslant \varepsilon\big\} = \mathscr{M}\big\{\gamma\colon \|\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\big\}, \text{ or}$$

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\big\{\|\zeta_{m+i,n+j} - \zeta\| \geqslant \varepsilon\big\} =$$
$$= \lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \mathscr{M}\big\{\gamma\colon \|\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\| \geqslant \varepsilon\big\} =$$
$$= \frac{1}{2^{k_1+k_2}} \to 0, \text{ uniformly in } m, n.$$

Thus, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ in measure. Further, we have

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} E\big[\|\zeta_{m+i,n+j}(\gamma) - \zeta(\gamma)\|\big] = \frac{1}{2^{k_1+k_2}} \to 0, \text{uniformly in } m, n.$$

Therefore, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is also strongly almost convergent to ζ in mean.

However, for any $\gamma([0,1])$, there is an infinite number of intervals of the form $[\frac{t}{2^{k_1+k_2}}, \frac{t+1}{2^{k_1+k_2}}]$ containing γ. Thus, the double sequence $\{\zeta_{m,n}\}$ is not convergent to 0. In other words, the double sequence $\{\zeta_{m,n}\}$ is not strongly almost convergent with respect to almost surely convergence to γ.

Proposition 1. *The complex uncertain double sequence* $\{\zeta_{m,n}\}$ *is strongly almost convergent to* ζ *with respect to almost surely convergence if and only if for any* $\varepsilon > 0$ *and uniformly in* m, n *we have*

$$\mathscr{M}\Big(\bigcap_{i=j=1}^{\infty}\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}(\gamma)-\zeta(\gamma)\|\geqslant\varepsilon\Big\}\Big)=0.$$

Proof. By the definition of strongly almost convergence with respect to almost surely convergence of complex uncertain double sequence, for all $\gamma \in \Lambda$ there exists an event with $\mathscr{M}\{\Lambda\} = 1$, such that

$$\lim_{p,q\to\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}(\gamma)-\zeta(\gamma)\|=0, \text{ uniformly in } m,n.$$

Then, for any $\varepsilon > 0$ such that for any event $\gamma \in \Lambda$ we have

$$\mathscr{M}\Big(\bigcup_{i=j=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|<\varepsilon\Big)=1, \text{ uniformly in } m,n.$$

It follows from the duality axiom of the uncertain measure that

$$\mathscr{M}\Big(\bigcap_{i=j=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big)=0, \text{ uniformly in } m,n.$$

Thus, the proposition is proved. □

Proposition 2. *Let* $\{\zeta_{m,n}\}$ *be sequences of complex uncertain variables, where* $m, n = 1, 2, 3\ldots$ *Then the complex uncertain double sequence* $\{\zeta_{m,n}\}$ *is strongly almost convergent to* ζ *with respect to uniformly almost surely convergence if and only if,*

$$\lim_{p,q\to\infty}\mathscr{M}\Big(\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big)=0, \textit{ uniformly in } m,n.$$

Proof. If the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent with respect to uniformly almost surely convergence to ζ, then for any $\delta > 0$ there exist a B, such that $\mathscr{M}(B) < \delta$ and the double sequence $\{\zeta_{m,n}\}$ strongly almost uniformly converges to ζ on $\Gamma - B$. We have

$$\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| < \varepsilon, \text{ uniformly in } m,n.$$

That is,

$$\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \varepsilon\Big\} \subset B.$$

It follows from the sub-additivity axiom of uncertain measure that

$$\mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \varepsilon\Big\}\Big) \leqslant \mathscr{M}\{B\} < \delta.$$

This implies:

$$\lim_{p,\,q\to\infty}\mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \varepsilon\Big\}\Big) = 0,$$

uniformly in m,n. Conversely, let,

$$\lim_{p,\,q\to\infty}\mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \varepsilon\Big\}\Big) = 0,$$

uniformly in m,n.

Then, for any $\delta > 0$, $k \geqslant 1$, and uniformly in m,n there exist k_t for which

$$\mathscr{M}\Big(\bigcup_{m=k_t}^{\infty}\bigcup_{n=k_t}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \frac{1}{k}\Big\}\Big) < \frac{\delta}{2^k}.$$

Consider, $B = \bigcup_{m=k_t}^{\infty}\bigcup_{n=k_t}^{\infty}\big\{\|\zeta_{m+i,n+j}-\zeta\| \geqslant \frac{1}{k}\big\}$. Then, for uniform values of m,n:

$$\mathscr{M}\{B\} \leqslant \mathscr{M}\Big(\bigcup_{m=k_t}^{\infty}\bigcup_{n=k_t}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\| \geqslant \frac{1}{k}\Big\}\Big) \leqslant \delta.$$

However, $\sup\limits_{\gamma\in\Gamma-B} \|\zeta_{m,n}-\zeta\| < \frac{1}{k}$ for any $k=1,2,3\ldots$, and $m,n>k_t$. Hence, the proposition is proved. □

Theorem 5. *If the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ with respect to uniformly almost surely convergence, then $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ with respect to almost surely convergence.*

Proof. It follows from Proposition 2 that if the complex uncertain double sequence$\{\zeta_{m,n}\}$ is strongly almost convergent with respect to uniformly almost surely convergence to ζ, then

$$\lim_{p,\,q\to\infty} \mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\Big)=0,$$

for uniformly in m,n. Since,

$$\mathscr{M}\Big(\bigcap_{i=j=1}^{\infty}\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\Big)\leqslant$$
$$\leqslant\mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\Big).$$

Taking limits as $p,q\to\infty$ in both sides, we obtain

$$\lim_{p,q\to\infty} \mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}||\zeta_{m+i,n+j}-\zeta||\geqslant\varepsilon\Big\}\Big)=0,$$

uniformly in m,n.

By Proposition 1, the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ. □

Theorem 6. *If the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ with respect to uniformly almost surely convergence, then $\{\zeta_{m,n}\}$ is strongly almost convergent to ζ in measure.*

Proof. If the complex uncertain double sequence $\{\zeta_{m,n}\}$ is strongly almost convergent with respect to uniformly almost surely convergence to ζ, then, from Proposition 2, we have

$$\lim_{p,\,q\to\infty} \mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\Big)=0,$$

uniformly in m, n and

$$\mathscr{M}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\leqslant$$
$$\leqslant\mathscr{M}\Big(\bigcup_{m=i}^{\infty}\bigcup_{n=j}^{\infty}\Big\{\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\|\zeta_{m+i,n+j}-\zeta\|\geqslant\varepsilon\Big\}\Big).$$

Taking limits as $p, q \to \infty$, we obtain that the double sequence $\{\zeta_{m,n}\}$ converges to ζ in measure. □

4. Conclusion. In this article, we have established that if a complex uncertain double sequence is strongly almost convergent in mean, it is convergent in measure and distribution also. Moreover, if it is strongly almost convergent with respect to uniformly almost surely convergence, it is convergent with respect to almost surely convergence as well. These concepts can also be generalized and extended in different directions.

References

[1] Bromwich T. J. I'A. *An Introduction to the Theory of Infinite Series.* MacMillan & Co. Ltd, New York, 1965.

[2] Chen X., Liu B. *Existence and uniqueness theorem for uncertain differential equations.* Fuzzy Optim. Decis. Mak., 2010, vol. 9, pp. 69 – 81.

[3] Chen X., Ning Y., Wang X. *Convergence of complex uncertain sequences.* J. Intell. & Fuzzy Syst., 2016, vol. 30, pp. 3357 – 3366.

[4] Chen X., Ralescu D. A. *B-spline method of uncertain statistics with applications to estimate travel distance.* J. of Uncertain Syst., 2012, vol. 6, pp. 256 – 262.

[5] Das B., Tripathy B. C., Debnath P., Bhattacharya B. *Characterization of statistical convergence of complex uncertain double sequence.* Anal. Math. Phy. 2020, vol. 10, pp. 1 – 20.

[6] Das B., Tripathy B. C., Debnath P., Nath J., Bhattacharya B. *Almost Convergence of Complex Uncertain Triple Sequences.* Proc. Nat. Acad. Scie., India Section A: Phy. Scie., 2021.
DOI: https://doi.org/10.1007/s40010-020-00721-w

[7] Hardy G. H. *On the convergence of certain multiple series.* Proc. Camb. Phil. Soc., 1917, vol. 19, pp. 86 – 95.

[8] King J. *Almost summable sequences.* Proc. Amer. Math. Soc., 1966, vol. 17, pp. 1219 – 1225.

[9] Liu B. *Uncertain entailment and modus ponens in the framework of uncertain logic*. J. of Uncertain Syst., 2009, vol. 3, pp. 243 – 251.

[10] Liu B. *Uncertain set theory and uncertain inference rule with application to uncertain control*. J. of Uncertain Syst., 2010, vol. 4, pp. 83 – 98.

[11] Liu B. *Uncertain logic for modelling human language*. J. of Uncertain Syst., 2011, vol. 5, pp. 3 – 20.

[12] Liu B. *Uncertainty Theory (5th edition)*. Springer-Verlag, Berlin, 2016.

[13] Lorentz G. G. *A contribution to the theory of divergent sequences*. Acta Math., 1948, vol. 80, pp. 167 – 190.

[14] Maddox I. *On strong almost convergence*. Math. Proc. Camb. Phil. Soc., 1979, vol. 85, pp. 345 – 350.

[15] Merriman M. *A set of necessary and sufficient conditions for the Cesàro summability of double series*. Annals Math., 1928, vol. 29, pp. 343 – 354.

[16] Moricz F., Rhoades B. E. *Almost convergence of double sequences and strong regularity of summability matrices*. Math. Proc. Camb. Phil. Soc., 1988, vol. 104, pp. 283 – 294.

[17] Mursaleen M. *Absolute almost convergent sequences*. Houston Jour. Math., 1984, vol. 10, pp. 427 – 431.

[18] Nath J., Tripathy B. C., Debnath P., Bhattacharya B. *Strongly Almost Convergence in Sequences of Complex Uncertain Variables*. Comm. Statist. Theory Methods., 2021.
DOI: https://doi.org/10.1080/03610926.2021.1921802

[19] Peng Z. *Complex Uncertain variable*. Doctoral Dissertation, Tsinghua University, 2012.

[20] Prngsheim A. *Zur theorie der zweifach unendlichen Zahlenfolgen*. Math. Ann., 1900, vol. 53, pp. 289 – 321.

[21] Savas E. *Strong almost convergence and almost λ-statistical convergence*. Hokkaido Math. Jour., 2000, vol. 29, pp. 531 – 536.

[22] Tripathy B. C. *On Double Sequences*. Bullt. Marathwada Math. Soc., 2019, vol. 20, pp. 33 – 49.

[23] Tripathy B. C., Datta D. *Convergence of complex uncertain double sequences*. New Math. Nat. Comp., 2020, vol. 16, pp. 447 – 459.

[24] You C. *On the convergence of uncertain sequences*. Math. Comput. Modell., 2009, vol. 49, pp. 482 – 487.

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