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ON GENERALIZATIONS OF INTEGRAL INEQUALITIES

Abstract. In the present study, several new generalized integral inequalities of the Hadamard and Simpson-type are obtained. The results were obtained for functions whose first and third derivatives are either convex or satisfy the Lipschitz condition or the conditions of the Lagrange theorem. In a particular case, these results not only confirm but also improve some upper bounds, well known in the literature for the Simpson and Hermite-Hadamard-type inequalities.

Key words: *convex function, Hermite–Hadamard inequality, Simpson-type inequality, Lipschitz conditions, Lagrange theorem, Riemann–Liouville fractional integral*

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1. Introduction. One of the notions that attracts more attention in mathematics today is probably that of the convex function. Its theoretical repercussions and multiple applications have made it the center of multiple works and research. Many of these continuous extensions and generalizations can be found in [23].

Definition 1. *The function $\varphi : [\sigma, \nu] \to \mathbb{R}$ is said to be convex if inequality*

$$\varphi(\lambda x + (1-\lambda) y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

holds $\forall\ x, y \in [\sigma, \nu]$ and $\lambda \in [0, 1]$.

Definition 2. *A function $\varphi : [\sigma, \nu] \to \mathbb{R}$ is said to be quasi-convex if the inequality*

$$\varphi(\lambda x + (1-\lambda)y) \leqslant \max\{\varphi(x), \varphi(y)\}$$

holds $\forall\ x, y \in [\sigma, \nu]$ and $\lambda \in [0, 1]$.

One of the most important inequalities is called the Hermite – Hadamard inequality ([12], [11]):

$$\varphi\Big(\frac{\sigma+\nu}{2}\Big) \leqslant \frac{1}{\nu-\sigma}\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi \leqslant \frac{\varphi(\sigma)+\varphi(\nu)}{2}, \tag{1}$$

which is valid for any function φ, convex on the interval $[\sigma,\nu]$.

Along with the Hermite – Hadamard type of inequality, a well-known Simpson type of inequality is provided in the literature as follows: if $\varphi\colon [\sigma,\nu]\to\mathbb{R}$ is a four times continuously differentiable function on (σ,ν) and $\big\|\varphi^{(4)}\big\|_{\infty} := \sup\limits_{x\in(\sigma,\nu)}\big|\varphi^{(4)}(x)\big| < \infty$, then

$$\left|\frac{\nu-\sigma}{3}\Big[\frac{\varphi(\sigma)+\varphi(\nu)}{2}+2\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]-\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi\right| \leqslant \frac{(\nu-\sigma)^5}{2880}\,\big\|\,\varphi^{(4)}\big\|_{\infty}. \tag{2}$$

In the last few decades, many researchers in the field of inequalities have refined, extended, and obtained new inequalities of the Hadamard and Simpson types for various classes of convex functions; for example, see [4], [6], [7], [10], [13], [24] and the references therein. A number of previous studies were devoted to obtaining new inequalities of the Simpson type using derivatives of a lower order than the fourth (for example, in [4], [10]). In addition, in many previous articles, Simpson-type inequalities were refined and obtained for various classes of convexity of functions. For example, see Alomari and Hussain in [2]. For quasi-convex functions, Hussain and Qaisar in [14] obtained the inequalities through preinvexity and prequasiinvexity. See Hua et al. in [13] and Chun et al. in [7] for s-convex and Bayraktar in [4] for r-convex functions: these authors obtained inequalities by using special means. Butt et al. in [6] also obtained Hadamard-type inequalities for η-quasi-convex and s-Godunov-Levin convex functions in terms of fractional integral operators. Özdemir et al. [24] obtained new integral inequalities for $(\alpha,s,m)-$convex functions. The following three inequalities are known well in the literature. In [26] (Corollary 1), Sarikaya et al. proved the following inequality:

$$\left|\frac{\nu-\sigma}{3}\Big[\frac{\varphi(\sigma)+\varphi(\nu)}{2}+2\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]-\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi\right| \leqslant$$

$$\leqslant \frac{5\,(\nu-\sigma)^2\,[|\ \varphi'(\sigma)|+|\ \varphi'(\nu)|]}{72}. \tag{3}$$

The estimate (3) was confirmed in some recent articles (see, for example, Corollary 2.5 in [16] for $w=1$) as a special case of the obtained result. Dragomir and Agarwal in [9] (Theorem 2.2) proved the following inequality:

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2}-\frac{1}{\nu-\sigma}\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi\right|\leqslant\frac{(\nu-\sigma)\,[|\varphi'\,(\sigma)|+|\varphi'\,(\nu)|]}{8}. \tag{4}$$

This inequality was confirmed in some recent papers (see, for example Corollary 3.1(4) in [20].In [17] (Theorem 2.2), Kirmaci obtained the following inequality:

$$\left|\frac{1}{\nu-\sigma}\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi-\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right|\leqslant\frac{(\nu-\sigma)\,[|\varphi'\,(\sigma)|+|\varphi'\,(\nu)|]}{8}. \tag{5}$$

The inequality (5) was confirmed in some recent articles (see, for example, Remark 2 in [27] and Remark 2 in [5]). The classical definition of a Riemann – Liouville fractional integral in the literature is given in the following way:

Definition 3. *Let $\varphi\in L[\sigma,\nu]$. The Riemann – Liouville integrals $J^{\alpha}_{\sigma+}\varphi$ and $J^{\alpha}_{\nu-}\varphi$ of order $\alpha>0$ with $\sigma\geqslant 0$ are defined by*

$$J^{\alpha}_{\sigma+}\varphi\ (x)=\frac{1}{\Gamma(\alpha)}\int\limits_{\sigma}^{x}(x-\xi)^{\alpha-1}\,\varphi(\xi)d\xi,\qquad x>\sigma,$$

$$J^{\alpha}_{\nu-}\varphi\ (x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\nu}(\xi-x)^{\alpha-1}\,\varphi(\xi)d\xi,\qquad x<\nu,$$

respectively, where $\Gamma(\alpha)=\int\limits_{0}^{\infty}e^{-u}u^{\alpha-1}du$. Here $J^{0}_{\sigma+}\varphi(x)=J^{0}_{\nu-}\varphi(x)=\varphi(x)$. In the case of $\alpha=1$, the fractional integral reduces to the classical integral.

This study was motivated by the work [8], in which the authors obtained an estimate of the Hadamard-type midpoint inequality in terms of the Lipschitz constant.

The aim of the article is to construct new generalized integral inequalities using the first and the third derivatives of a function and, using these inequalities, to obtain inequalities of the Hadamard and Simpson types by taking into account the fact that the derivatives of the function satisfy the conditions of either convexity, or satisfy the Lipschitz condition, or the conditions of the Lagrangian theorem.

1. Results obtained by using the first derivative. For the sake of brevity, we will use the following notation for some expressions:

$$\mathbf{U}(J^{\alpha}_{\sigma+}, J^{\alpha}_{\nu-}) := \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\nu-\sigma)^{\alpha}}\Big[J^{\alpha}_{\sigma+}\varphi\Big(\frac{\sigma+\nu}{2}\Big) + J^{\alpha}_{\nu-}\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big],$$

$$\mathbf{\Psi}(w,\varphi) := \int_0^1 w'(\xi)\Big[\varphi\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big) + \varphi\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big)\Big]d\xi.$$

Lemma 1. *Let $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ and $\varphi \in C^1(I^{\circ})$ (I° be the interior of I) and $\sigma, \nu \in I$ with $0 \leqslant \sigma < \nu < \infty$. If $\varphi' \in L_1[\sigma,\nu]$ and $w(\xi)$ is a continuous function on the interval $[0,1]$, then we have*

$$\Big[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2} - w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big] - \frac{1}{2}\mathbf{\Psi}(w,\varphi) = \frac{\nu-\sigma}{4}\left(I_1 - I_2\right), \quad (6)$$

where

$$I_1 = \int_0^1 w(\xi)\varphi'\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big)d\xi,$$

$$I_2 = \int_0^1 w(\xi)\varphi'\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big)d\xi.$$

Proof. Integrating by parts for I_1 and I_2, we get

$$I_1 = \frac{2[w(1)\varphi(\nu) - w(0)\varphi(\frac{\nu+\sigma}{2})]}{\nu-\sigma} - \frac{2}{\nu-\sigma}\int_0^1 w'(\xi)\varphi\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big)d\xi,$$

$$I_2 = \frac{2[w(0)\varphi(\frac{\nu+\sigma}{2}) - w(1)\varphi(\sigma)]}{\nu-\sigma} + \frac{2}{\nu-\sigma}\int_0^1 w'(\xi)\varphi\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big)d\xi.$$

By subtracting the second equality from the first, dividing both sides of the resulting equality by expression A, and taking into account the accepted designations, we obtain equality (6). The proof is completed. □

Remark. *By choosing the function $w(\xi)$ in the left-hand side of (6), we can get various Hermite – Hadamard or Simpson-type expressions, for example:*

1) *If $w(\xi) = c\xi + d$ and $((d < 0$ and $c > |d|)$ or $(c < 0$ and $|c| > d > 0))$ then, in the left-hand side of identity (6), we get an expression of the Simpson type:*

$$\frac{\nu-\sigma}{c}\Big[\,(c+d)\cdot\frac{\varphi(\sigma)+\varphi(\nu)}{2}-d\cdot\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]-\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi = \\ = \frac{(\nu-\sigma)^2}{4c}\left(I_1-I_2\right).$$

2) *If $w(\xi) = c\xi$, we get the identity obtained by Alomari in [1] (Lemma 2.1), Iqbal in [15] (Lemma 2, for $\lambda = \mu$), Latif in [18] (Lemma 2.1), and in [19] (Lemma 1, for $x = \sigma$ or $x = \nu$).*
3) *If $c \cdot d < 0$ and $|d| = |c|$, we get the identity obtained by Iqbal in [15] (Lemma 3, for $\lambda = \mu$);*
4) *If $w(\xi) = c\xi^\alpha + d$, then, under one of the conditions $(d < 0\ \ \&\ \ c > |d|)$ or $(c < 0\, \& \, |c| > d)$ on the left-hand side of (6), we will always have an expression like Simpson by using the fractional operator*

$$\frac{1}{c}\Big[\,(c+d)\cdot\frac{\varphi(\sigma)+\varphi(\nu)}{2}+d\cdot\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]-\mathbf{U}(J^{\alpha}_{\sigma+},J^{\alpha}_{\nu-}) = \\ = \frac{\nu-\sigma}{4c}\left(I_1-I_2\right). \quad (7)$$

5) *If $w(\xi) = \frac{\xi^\alpha}{2} - \frac{1}{3}$, we get the identity obtained by Matloka in [21] (Lemma 5);*
6) *If $w(\xi) = c\xi^\alpha$, we get the identity obtained by Mihai in [22](Lemma 1, for $x = \nu$) and Awan et al. in [3] (Lemma 2.5, for $n = 1$ and $x = \nu$).*
7) *If $c \cdot d < 0$ and $|d| = |c|$, we get the identity, which is equivalent to the identity obtained by Sarıkaya in [27](Lemma 3).*

8) *If* $w(\xi) = (1-\xi)^{\alpha}$, *we get the identity obtained by Sarıkaya and Yıldırım in [27](Lemma 3).*

Theorem 1. *Let* $\varphi : I \to \mathbb{R}$ *and* $\varphi \in C^1(I^\circ)$. *For* $0 \leqslant \sigma < \nu$, *suppose that* $\sigma, \nu \in I^\circ$ *and* $|\varphi'|$ *is convex on* $[\sigma, \nu]$. *Then the equality*

$$\left|\left[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2} - w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right] - \frac{1}{2}\mathbf{\Psi}(w,\varphi)\right| \leqslant$$
$$\leqslant \frac{(\nu-\sigma)\left[|\varphi'(\sigma)| + |\varphi'(\nu)|\right]}{4}\int\limits_0^1 |w(\xi)|d\xi \quad (8)$$

holds. Here $\mathbf{\Psi}(w,\varphi)$ *has been defined above.*

Proof. Since $|\varphi'|$ is convex, then, from (6) for $(I_1 - I_2)$, we can write:

$$|I_1 - I_2| \leqslant |I_1| + |I_2| \leqslant \int\limits_0^1 |w(\xi)| \left[\left|\varphi'\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big)\right| + \right.$$
$$\left. + \left|\varphi'\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big)\right|\right] d\xi = \left[|\varphi'(\sigma)| + |\varphi'(\nu)|\right]\int\limits_0^1 |w(\xi)|\, d\xi.$$

The proof of the theorem follows from this inequality. □

Remark. *If we accept that* $w(\xi) = \frac{1}{3} - \frac{\xi}{2}$, *then, from (8), we obtain (3). If we choose* $w(\xi) = c\xi$ *with* $c \neq 0$, *then, from (8), we obtaine the inequality (4). If* $w(\xi) = c(1-\xi)$ *with* $c \neq 0$, *then, from (8), we obtaine the inequality (5). Take* $w(\xi) = \frac{\xi^{\alpha}}{2} - \frac{1}{3}$ *and take into account* $\frac{\xi^{\alpha}}{2} - \frac{1}{3} \leqslant \frac{1}{3}$, $\forall \xi \in [0,1]$; *then, from (8), we get the inequality obtained by Matloka in [21] (see Corollary 7).*

Remark. *If* $w(\xi) = c\xi^{\alpha}$, *then (8) gives for* $c = 2$ *the inequality obtained by Mihai in [22] (Theorem 1, for* $x = \nu$*) and Set et al. in [28] (Theorem 8, for* $n = 1$, $x = \nu$, $m = 1$*).*

Remark. *If* $w(\xi) = 1 - \xi^{\alpha}$, *then, from (8), we get*

$$\left|\varphi\Big(\frac{\sigma+\nu}{2}\Big) - 2\mathbf{U}(J^{\alpha}_{\sigma+}, J^{\alpha}_{\nu-})\right| \leqslant \frac{\alpha(\nu-\sigma)}{2(\alpha+1)}\left[|\varphi'(\sigma)| + |\varphi'(\nu)|\right].$$

Remark. *If* $w(\xi) = (1-\xi)^{\alpha}$, *then, from (8), we get*

$$\left|\varphi\Big(\frac{\sigma+\nu}{2}\Big) - \frac{2^{\alpha}\Gamma(\alpha+1)}{(\nu-\sigma)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\sigma+\nu}{2}\right)^{+}}\varphi(\nu) + J^{\alpha}_{\left(\frac{\sigma+\nu}{2}\right)^{-}}\varphi(\sigma)\right]\right| \leqslant$$
$$\leqslant \frac{\nu-\sigma}{4(\alpha+1)}\left[|\varphi'(\sigma)| + |\varphi'(\nu)|\right].$$

For $\alpha = 1$, *this inequality gives (5).*

Corollary 1. *Let* $\varphi : I = [\sigma,\nu] \to \mathbb{R}$ *and* $\varphi \in C^{1}(I^{\circ})$. *If* φ' *satisfies the Lipschitz condition on* I *with respect to* K, *then the following inequality holds:*

$$\left|\left[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2} - w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right] - \frac{1}{2}\mathbf{\Psi}(w,\varphi)\right| \leqslant$$
$$\leqslant \frac{K\cdot(\nu-\sigma)^{2}}{4}\int_{0}^{1}\xi\cdot|w(\xi)|\,d\xi, \quad (9)$$

where $\mathbf{\Psi}(w,\varphi)$ *was previously defined.*

Proof. For the $I_2 - I_1$ from (6) and by taking into account the Lipschitz condition, we can write

$$|I_2 - I_1| \leqslant \int_{0}^{1}|w(\xi)|\left|\varphi'\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big) - \varphi'\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big)\right|d\xi \leqslant$$
$$\leqslant K\int_{0}^{1}|w(\xi)|\left|\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu - \frac{1+\xi}{2}\sigma - \frac{1-\xi}{2}\nu\right|d\xi =$$
$$= K\cdot(\nu-\sigma)\int_{0}^{1}\xi\cdot|w(\xi)|\,d\xi.$$

This inequality obviously implies (9). □

Remark. *If we take into account the fact that the functions* φ' *are Lipschitz with the constant* $K \leqslant \underset{x\in[\sigma,\nu]}{m\sigma x}|\varphi''(x)| = \|\varphi''\|_{\infty} < \infty$, *then*

$$\left|\left[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2} - w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right] - \frac{1}{2}\mathbf{\Psi}(w,\varphi)\right| \leqslant$$

$$\leqslant \frac{(\nu-\sigma)^2 \|\varphi''\|_\infty}{4} \int_0^1 \xi \cdot |w(\xi)|\, d\xi. \quad (10)$$

Remark. *If we accept that* $w(\xi) = \frac{1}{3} - \frac{\xi}{2}$, *then from (9) and (10) we get the inequality obtained by Delavar et al. in [25] (see Remark* 1(2)*).*

Remark. *If* $w(\xi) = \xi$, *then from (9) and (10) we obtain the inequality*

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2} - \frac{1}{\nu-\sigma}\int_\sigma^\nu \varphi(\xi)d\xi\right| \leqslant \frac{K\cdot(\nu-\sigma)^2}{12} \leqslant \frac{(\nu-\sigma)^2\|\varphi''\|_\infty}{12}.$$

Remark.
(i0) If $w(\xi) = 1-\xi$, *then we get the inequality obtained by Delavar and Dragomir in [8] (Corollary* 2.4*).*
(i1) If we take $w(\xi) = \frac{\xi^\alpha}{2} - \frac{1}{3}$ *and take into account* $\frac{\xi^\alpha}{2} - \frac{1}{3} \leqslant \frac{1}{3}, \forall \xi \in [0,1]$, *then from (9) we obtain the inequality*

$$\left|\frac{1}{3}\left[\frac{\varphi(\sigma)+\varphi(\nu)}{2} + 2\varphi\left(\frac{\sigma+\nu}{2}\right)\right] - \mathbf{U}(J^\alpha_{\sigma+}, J^\alpha_{\nu-})\right| \leqslant$$
$$\leqslant \frac{K\cdot(\nu-\sigma)^2}{2}\int_0^1 \xi\cdot|w(\xi)|\,d\xi \leqslant \frac{K\cdot(\nu-\sigma)^2}{12}.$$

(i2) If $w(\xi) = c\xi^\alpha$, *then from (9) we get*

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2} - 2\mathbf{U}(J^\alpha_{\sigma+}, J^\alpha_{\nu-})\right| \leqslant \frac{|c|\, K\cdot(\nu-\sigma_1)^2}{4\,(\alpha+2)}.$$

(i3) If $w(\xi) = 1-\xi^\alpha$, *then from (9) we get*

$$\left|\varphi\left(\frac{\sigma+\nu}{2}\right) - 2\mathbf{U}(J^\alpha_{\sigma+}, J^\alpha_{\nu-})\right| \leqslant \frac{\alpha K\cdot(\nu-\sigma)^2}{8\,(\alpha+2)}.$$

(i4) If $w(\xi) = (1-\xi)^\alpha$, *then from (9) we get*

$$\left|\varphi\left(\frac{\sigma+\nu}{2}\right) - \frac{2^\alpha\Gamma(\alpha+1)}{(\nu-\sigma)^\alpha}\left[J^\alpha_{\left(\frac{\sigma+\nu}{2}\right)+}\varphi(\nu) + J^\alpha_{\left(\frac{\sigma+\nu}{2}\right)-}\varphi(\sigma)\right]\right| \leqslant \frac{K\cdot(\nu-\sigma)^2}{4(\alpha+1)(\alpha+2)}.$$

Theorem 2. *Let $\varphi : I \to \mathbb{R}$ and $\varphi \in C^1(I^\circ)$. For $0 \leqslant \sigma < \nu$; suppose that $\sigma, \nu \in I^\circ$ and φ' is a continuous function on the closed interval $[\sigma, \nu]$ and differentiable on the open interval (σ, ν). Then there exists some ς in (σ, ν), such as:*

$$\left| \left[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2} - w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big) \right] - \frac{1}{2}\boldsymbol{\Psi}(w,\varphi) \right| \leqslant$$

$$\leqslant \frac{(\nu-\sigma)^2 \left|\varphi''(\varsigma)\right|}{4} \int_0^1 \xi |w(\xi)| d\xi \leqslant \frac{(\nu-\sigma)^2 \left\|\varphi''\right\|_\infty}{4} \int_0^1 \xi |w(\xi)| d\xi, \quad (11)$$

where $\boldsymbol{\Psi}(w,\varphi)$ has been previously defined and $\|\varphi''\|_\infty = \max\limits_{x\in[\sigma,\nu]} |\varphi''(x)|$.

Proof. From (6) and Lagrange's theorem, we can write

$$|I_2 - I_1| \leqslant \int_0^1 |w(\xi)| \left| \varphi'\Big(\frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu\Big) - \varphi'\Big(\frac{1+\xi}{2}\sigma + \frac{1-\xi}{2}\nu\Big) \right| d\xi \leqslant$$

$$\leqslant |\varphi''(\varsigma)| (\nu-\sigma) \int_0^1 |w(\xi)| \left| \frac{1-\xi}{2}\sigma + \frac{1+\xi}{2}\nu - \frac{1+\xi}{2}\sigma - \frac{1-\xi}{2}\nu \right| d\xi =$$

$$= |\varphi''(\varsigma)| (\nu-\sigma) \int_0^1 \xi \cdot |w(\xi)| \, d\xi.$$

This inequality obviously implies (11). □

Remark. *If we accept that $w(\xi) = \dfrac{1}{3} - \dfrac{\xi}{2}$, then from (11) we get*

$$\left| \frac{\nu-\sigma}{3} \left[\frac{\varphi(\sigma_2)+\varphi(\sigma)}{2} + 2\varphi\Big(\frac{\sigma+\nu}{2}\Big) \right] - \int_\sigma^\nu \varphi(\xi) d\xi \right| \leqslant$$

$$\leqslant \frac{2\left|\varphi''(\xi)\right| \cdot (\nu-\sigma)^3}{81} \leqslant \frac{2\left\|\varphi''\right\|_\infty \cdot (\nu-\sigma)^3}{81}, \quad (12)$$

Remark. *If we take $w(\xi) = \dfrac{\xi^\alpha}{2} - \dfrac{1}{3}$ and take into account $\dfrac{\xi^\alpha}{2} - \dfrac{1}{3} \leqslant \dfrac{1}{3}$, $\forall \xi \in [0,1]$, then from (11) we obtain the inequality*

$$\left|\frac{1}{3}\left[\frac{\varphi(\sigma)+\varphi(\nu)}{2}+2\varphi\left(\frac{\sigma+\nu}{2}\right)\right]-\mathbf{U}(J^{\alpha}_{\sigma+},J^{\alpha}_{\nu-})\right|\leqslant$$
$$\leqslant\frac{(\nu-\sigma)^2\,|\varphi''(\xi)|}{12}\leqslant\frac{\|\varphi''\|_\infty\cdot(\nu-\sigma)^3}{6}.$$

Remark. *If* $w(\xi)=c\xi^\alpha$, *then from (11) we get*

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2}-2\mathbf{U}(J^{\alpha}_{\sigma+},J^{\alpha}_{\nu-})\right|\leqslant\frac{|c||\varphi''(\xi)|(\nu-\sigma)^2}{4(\alpha+2)}\leqslant\frac{|c|(\nu-\sigma)^2\|\varphi''\|_\infty}{4(\alpha+2)}.$$

Remark. *If* $w(\xi)=1-\xi^\alpha$, *then from (11) we get*

$$\left|\varphi\left(\frac{\sigma+\nu}{2}\right)-2\mathbf{U}(J^{\alpha}_{\sigma+},J^{\alpha}_{\nu-})\right|\leqslant\frac{\alpha\,(\nu-\sigma)^2\,|\varphi''(\xi)|}{8\,(\alpha+2)}\leqslant\frac{\alpha\,(\nu-\sigma)^2\,\|\varphi''\|_\infty}{8\,(\alpha+2)}.$$

Remark. *If* $w(\xi)=(1-\xi)^\alpha$, *then from (11), we get*

$$\left|\varphi\left(\frac{\sigma+\nu}{2}\right)-\frac{2^\alpha\Gamma(\alpha+1)}{(\nu-\sigma)^\alpha}\left[J^{\alpha}_{\left(\frac{\sigma+\nu}{2}\right)^+}\varphi\left(\nu\right)+J^{\alpha}_{\left(\frac{\sigma+\nu}{2}\right)^-}\varphi\left(\sigma\right)\right]\right|\leqslant$$
$$\leqslant\frac{(\nu-\sigma)^2\,|\varphi''(\xi)|}{4\,(\alpha+1)\,(\alpha+2)}\leqslant\frac{(\nu-\sigma)^2\,\|\varphi''\|_\infty}{4\,(\alpha+1)\,(\alpha+2)}.$$

It was established that for some functions φ, defined on the interval $[\sigma,\nu]$, for all $\xi\in[0,1]$, the following inequality holds:

$$|\varphi(\xi\sigma+(1-\xi)\nu)-\varphi(\xi\nu+(1-\xi)\sigma)|\leqslant|1-2\xi|\,|\varphi(\nu)-\varphi(\sigma)|\,. \tag{13}$$

For example, the function $\varphi(x)=x^n$, $x\in[0,\nu]$ satisfies the inequality (13) for all $n\in N$.

For many functions that are convex on a given interval, it is impossible to prove analytically whether (13) holds or does not hold, but it is easy to prove numerically by using the MS Excel spreadsheet software (for example: for functions $f(x)=-\ln(x+1)$, $g(x)=e^x$, $w(x)=\sin x+\cos x+x^2$ and $h(x)=\sqrt{x^2+1}$).

From (6), it is easy to establish the validity of the following corollary:

Corollary 1. *Let* $\varphi\colon I\to\mathbb{R}$ *and* $\varphi\in C^1(I^\circ)$. *For* $0\leqslant\sigma<\nu$, *suppose that* σ, $\nu\in I^\circ$ *and* φ' *satisfies the inequality (13) on* $[\sigma,\nu]$. *Then the inequality*

$$\left|\left[w(1)\frac{\varphi(\sigma)+\varphi(\nu)}{2}-w(0)\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right]-\frac{1}{2}\mathbf{\Psi}(w,\varphi)\right|\leqslant$$
$$\leqslant\frac{(\nu-\sigma)\left|\varphi'(\nu)-\varphi'(\sigma)\right|}{4}\int\limits_0^1\xi\cdot|w(\xi)|\,d\xi \quad (14)$$

holds. Here, $\mathbf{\Psi}(w,\varphi)$ *has been previously defined.*

Proof. By taking into account the inequality (13) for the right-hand side of (6), we can write

$$|I_2-I_1|\leqslant\int\limits_0^1|w(\xi)|\left|\varphi'\Big(\frac{1-\xi}{2}\nu+\frac{1+\xi}{2}\sigma\Big)-\varphi'\Big(\frac{1+\xi}{2}\nu+\frac{1-\xi}{2}\nu\Big)\right|d\xi\leqslant$$
$$\leqslant\int\limits_0^1|w(\xi)|\left|\frac{1-\xi}{2}\varphi'(\nu)+\frac{1+\xi}{2}\varphi'(\sigma)-\frac{1+\xi}{2}\varphi'(\nu)-\frac{1-\xi}{2}\varphi'(\sigma)\right|d\xi=$$
$$=|\varphi'(\sigma)-\varphi'(\nu)|\int\limits_0^1\xi\cdot|w(\xi)|\,d\xi.$$

This inequality obviously implies (14). □

Remark. *If we accept that* $w(\xi)=\frac{1}{3}-\frac{\xi}{2}$, *then from (14) we get*

$$\left|\frac{\nu-\sigma}{6}\left[\varphi(\nu)+\varphi(\sigma)+4\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right]-\int\limits_\sigma^\nu\varphi(\xi)d\xi\right|\leqslant$$
$$\leqslant\frac{2\left(\nu-\sigma\right)^2}{81}\left|\varphi'(\nu)-\varphi'(\sigma)\right|. \quad (15)$$

Remark. *Obviously, the error of estimate (15) is more than 2,5 times better than the estimate (3).*

Remark. *If* $w(\xi)=\xi$, *then from (14) we obtain the inequality*

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2}-\frac{1}{\nu-\sigma}\int\limits_\sigma^\nu\varphi(\xi)d\xi\right|\leqslant\frac{\nu-\sigma}{12}\left|\varphi'(\nu)-\varphi'(\sigma)\right|. \quad (16)$$

Remark. *If $w(\xi)=1-\xi$, then from (14) we obtain the inequality*

$$\left|\frac{1}{\nu-\sigma}\int\limits_{\sigma}^{\nu}\varphi(\xi)d\xi-\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right|\leqslant\frac{\nu-\sigma}{24}\left|\varphi'(\nu)-\varphi'(\sigma)\right|. \tag{17}$$

Remark. *If we take $w(\xi)=\dfrac{\xi^\alpha}{2}-\dfrac{1}{3}$ and by take into account $\dfrac{\xi^\alpha}{2}-\dfrac{1}{3}\leqslant\dfrac{1}{3}$, $\forall\xi\in[0,1]$, then from (14) we obtain the inequality*

$$\left|\frac{1}{3}\Big[\frac{\varphi(\sigma)+\varphi(\nu)}{2}+2\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]-\mathbf{U}(J^\alpha_{\sigma+},J^\alpha_{\nu-})\right|\leqslant$$
$$\leqslant\frac{\nu-\sigma}{12}\left|\varphi'(\nu)-\varphi'(\sigma)\right|.$$

Remark. *If $w(\xi)=c\xi^\alpha$, then from (14) we get*

$$\left|\frac{\varphi(\sigma)+\varphi(\nu)}{2}-2\mathbf{U}(J^\alpha_{\sigma+},J^\alpha_{\nu-})\right|\leqslant\frac{|c|\,(\nu-\sigma)\left|\varphi'(\nu)-\varphi'(\sigma)\right|}{4\,(\alpha+2)}.$$

For $\alpha=1$ and $|c|=1$, this inequality gives (17).

Remark. *If $w(\xi)=1-\xi^\alpha$, then from (14) we get*

$$\left|\varphi\Big(\frac{\sigma+\nu}{2}\Big)-2\mathbf{U}(J^\alpha_{\sigma+},J^\alpha_{\nu-})\right|\leqslant\frac{\alpha\left|\varphi'(\nu)-\varphi'(\sigma)\right|}{8\,(\alpha+2)}.$$

For $\alpha=1$, this inequalitys gives (17).

Remark. *If $w(\xi)=(1-\xi)^\alpha$, then from (14) we get*

$$\left|\varphi\Big(\frac{\sigma+\nu}{2}\Big)-2\mathbf{U}(J^\alpha_{\sigma+},J^\alpha_{\nu-})\right|\leqslant\frac{\left|\varphi'(\nu)-\varphi'(\sigma)\right|}{4\,(\alpha+1)\,(\alpha+2)}.$$

For $\alpha=1$, this inequality gives (17).

For convex functions, provided that inequality (13) holds, the obtained estimates (16) and (17) for the right and left Hermite–Hadamard inequalities are undoubtedly better than estimates (4) and (5) available in the literature.

2. Results of using the third derivative. The literature includes some studies where the inequality estimate is obtained on subintervals of the interval $[\sigma,\nu]$. The formulated lemma gives us an identity depending on the parameter h. This parameter allows us to get estimates on subintervals of the interval $[\sigma,\nu]$.

Lemma 2. *Let $\varphi\colon I\subset\mathbb{R}\to\mathbb{R}$, and $\varphi\in C^3(I^\circ)$, and $\sigma,\nu\in I$ with $0\leqslant\sigma<\nu<\infty$. If $\varphi'''\in L_1[\sigma,\nu]$ and $h\in(0,1]$, then we have:*

$$\left(\int\limits_{c}^{\frac{\nu+c}{2}}\varphi(\xi)d\xi+\int\limits_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi\right)-\frac{h(\nu-\sigma)}{3}\left[\frac{\varphi(d)+\varphi(c)}{2}+\varphi\Big(\frac{\sigma+d}{2}\Big)+\varphi\Big(\frac{c+\nu}{2}\Big)\right]+$$
$$+\frac{h^2(\nu-\sigma)^2}{24}\left[\varphi'\Big(\frac{c+\nu}{2}\Big)-\varphi'\Big(\frac{\sigma+d}{2}\Big)\right]=\frac{(\nu-\sigma)^4}{12}\left(I_1-I_2\right),\quad (18)$$

where $c=h\sigma+(1-h)\nu$, $d=(1-h)\sigma+h\nu$ and $c,d\in[\sigma,\nu]$,

$$I_1=\int\limits_0^{\frac{h}{2}}\xi^2(h-2\xi)\varphi'''((1-h+\xi)\,\sigma+(h-\xi)\nu)d\xi,$$
$$I_2=\int\limits_0^{\frac{h}{2}}\xi^2(h-2\xi)\varphi'''((h-\xi)\sigma+(1-h+\xi)\,\nu)d\xi.$$

Proof. By integrating both integrals by parts thrice, and, finally, by making the change of variables, we get:

$$I_1=\int\limits_0^{\frac{h}{2}}\xi^2(h-2\xi)\varphi'''((1-h+\xi)\,\sigma+(h-\xi)\nu)d\xi=$$
$$=\frac{h^2}{2(\nu-\sigma)^2}\varphi'\Big(\frac{\sigma+d}{2}\Big)+\frac{4h}{(\nu-\sigma)^3}\varphi\Big(\frac{\sigma+d}{2}\Big)+$$
$$+\frac{2h}{(\nu-\sigma)^3}\varphi(d)-\frac{12}{(\nu-\sigma)^4}\int\limits_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi.$$

$$I_2=\frac{h^2}{2(\nu-\sigma)^2}\varphi'\Big(\frac{c+\nu}{2}\Big)-\frac{4h}{(\nu-\sigma)^3}\varphi\Big(\frac{c+\nu}{2}\Big)-$$
$$-\frac{2h}{(\nu-\sigma)^3}\varphi\left(c\right)+\frac{12}{(\nu-\sigma)^4}\int\limits_{c}^{\frac{c+\nu}{2}}\varphi(\xi)d\xi.$$

By taking into account $\nu - \dfrac{\nu-\sigma}{2}h = \dfrac{c+\nu}{2}$ for the difference between the integrals, we obtain

$$I_1 - I_2 = \frac{h^2}{2(\nu-\sigma)^2}\Big[\varphi'\Big(\frac{\sigma+d}{2}\Big) - \varphi'\Big(\frac{c+\nu}{2}\Big)\Big] - \\ - \frac{4h}{(\nu-\sigma)^3}\Big[\frac{\varphi(d)+\varphi(c)}{2} + \Big(\varphi\Big(\frac{\sigma+d}{2}\Big) + \varphi\Big(\frac{c+\nu}{2}\Big)\Big)\Big] + \\ + \frac{12}{(\nu-\sigma)^4}\Big[\int\limits_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi + \int\limits_{c}^{\frac{c+\nu}{2}}\varphi(\xi)d\xi\Big].$$

By dividing both sides of this equation by the expression $\dfrac{(\nu-\sigma)^4}{12}$, we get (18). The proof is completed. □

Remark. *In (18), if we choose $h = 1$ and change the variables $1-2\xi = z$, then we get identity from [7] (see Lemma 2.1).*

Theorem 3. *Let $\varphi\colon I \to \mathbb{R}$ and $\varphi \in C^3(I^\circ)$. For $0 \leqslant \sigma < \nu$, suppose that $\sigma, \nu \in I^\circ$ and φ''' satisfies the Lipschitz condition on $[\sigma,\nu]$ with respect to K. Then the following Simpson-type inequality holds:*

$$\Big|\Big(\int\limits_{c}^{\frac{\nu+c}{2}}\varphi(\xi)d\xi + \int\limits_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi\Big) - \\ - \frac{h(\nu-\sigma)}{6}\Big[\varphi(d)+\varphi(c)+2\cdot\Big(\varphi\Big(\frac{\sigma+d}{2}\Big)+\varphi\Big(\frac{c+\nu}{2}\Big)\Big)\Big]\Big| + \\ + \Big|\frac{h^2(\nu-\sigma)^2}{24}\Big[\varphi'\Big(\frac{c+\nu}{2}\Big) - \varphi'\Big(\frac{\sigma+d}{2}\Big)\Big]\Big| \leqslant \\ \leqslant \frac{h^4\left(|7h-5|\right)(\nu-\sigma)^5\,K}{5760}, \quad (19)$$

where $h \in (0,1]$, $c = h\sigma + (1-h)\nu$ and $d = (1-h)\sigma + h\nu$.

Proof. For the $I_1 - I_2$ from (18) taking into account the Lipschitz condition, we can write:

$$|I_1 - I_2| \leqslant \int\limits_{0}^{\frac{h}{2}} \xi^2(h-2\xi)\,|\varphi'''((1-h+\xi)\,\sigma + (h-\xi)\nu) -$$

$$-\varphi'''((h-\xi)\sigma + (1-h+\xi)\,\nu)|\,d\xi \leqslant$$

$$\leqslant K\cdot(\nu-\sigma)\int_0^{\frac{h}{2}} \xi^2(h-2\xi)\,|2h-2\xi-1|\,d\xi.$$

Since $\forall\xi \in \left[0, \frac{h}{2}\right]$, $\begin{cases} 2h-2\xi-1<0, & h\in[0;0{,}5) \\ 2h-2\xi-1\geqslant 0, & h\in[0{,}5;1] \end{cases}$,

$$|I_1 - I_2| \leqslant K\cdot(\nu-\sigma)\left|\int\limits_0^{\frac{h}{2}} \xi^2(h-2\xi)\,(2\xi-2h+1)\,d\xi\right| =$$

$$= K\cdot(\nu-\sigma)\left|\int\limits_0^{\frac{h}{2}} \left[-(2h-1)\,h\xi^2 + 2\,(3h-1)\,\xi^3 - 4\xi^4\right] d\xi\right| =$$

$$= K\cdot(\nu-\sigma)\frac{h^4\,|7h-5|}{480}.$$

Multiplying both sides of the last inequality by the expression $\dfrac{(\nu-\sigma)^4}{12}$, we get (19). Proof is completed. □

Remark. *In (19), if we take* $h=\dfrac{1}{2}$, *we get*

$$\left|\int\limits_{\frac{3\sigma+\nu}{4}}^{\frac{3\nu+\sigma}{4}} \varphi(\xi)d\xi - \frac{\nu-\sigma}{6}\left[\varphi\Big(\frac{3\sigma+\nu}{4}\Big) + \varphi\Big(\frac{\nu+\sigma}{2}\Big) + \varphi\Big(\frac{\sigma+3\nu}{4}\Big)\right]\right| +$$

$$+\left|\frac{(\nu-\sigma)^2}{96}\left[\varphi'\Big(\frac{\sigma+3\nu}{4}\Big) - \varphi'\Big(\frac{3\sigma+\nu}{4}\Big)\right]\right| \leqslant \frac{(\nu-\sigma)^5\ \cdot K}{61400}. \quad (20)$$

Remark. *In (19), if we take* $h=1$, *then we obtain an analogue of the well-known inequality (2):*

$$\left|\int\limits_{\sigma}^{\nu} \varphi(\xi)d\xi - \frac{\nu-\sigma}{6}\left[\varphi\,(\sigma) + \varphi\,(\nu) + 4\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right]\right| \leqslant \frac{(\nu-\sigma)^5\ \cdot K}{2880}. \quad (21)$$

From (18) and Lagrange's theorem, it is not difficult (see the proof of Theorem 2) to prove the following theorem:

Theorem 4. *Let* $\varphi\colon I\to\mathbb{R}$, *and* $\varphi\in C^3(I^\circ)$. *For* $0\leqslant\sigma<\nu$, *suppose that* $\sigma,\nu\ \in I^\circ$ *and* φ''' *is a continuous function on the closed interval* $[\sigma,\nu]$

and differentiable on the open interval (σ,ν). *Then there exists some* $\varsigma\in(\sigma,\nu)$, *such as:*

$$\Bigg|\Bigg(\int_{c}^{\frac{\nu+c}{2}}\varphi(\xi)d\xi+\int_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi\Bigg)- -\frac{h(\nu-\sigma)}{6}\Big[\varphi(d)+\varphi(c)+2\cdot\Big(\varphi\Big(\frac{\sigma+d}{2}\Big)+\varphi\Big(\frac{c+\nu}{2}\Big)\Big)\Big]\Bigg|+ +\Big|\frac{h^2(\nu-\sigma)^2}{24}\Big[\varphi'\Big(\frac{c+\nu}{2}\Big)-\varphi'\Big(\frac{\sigma+d}{2}\Big)\Big]\Big|\leqslant \leqslant\frac{h^4\left|\varphi'(\varsigma)\right|\left(\left|7h-5\right|\right)(\nu-\sigma)^5}{5760},\qquad(22)$$

where $h\in(0,1]$, $c=h\sigma+(1-h)\nu$ *and* $d=(1-h)\sigma+h\nu$.

Theorem 5. *Let* $\varphi:I\to\mathbb{R}$ *and* $\varphi\in C^3(I^\circ)$. *For* $0\leqslant\sigma<\nu$, *suppose that* $\sigma,\nu\in I^\circ$ *and* $|\varphi'''|$ *is convex on* $[\sigma,\nu]$. *Then the inequality:*

$$\Bigg|\Bigg(\int_{c}^{\frac{\nu+c}{2}}\varphi(\xi)d\xi+\int_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi\Bigg)- -\frac{h(\nu-\sigma)}{6}\Big[\varphi(d)+\varphi(c)+2\cdot\Big(\varphi\Big(\frac{\sigma+d}{2}\Big)+\varphi\Big(\frac{c+\nu}{2}\Big)\Big)\Big]\Bigg|+ +\Big|\frac{h^2(\nu-\sigma)^2}{24}\Big[\varphi'\Big(\frac{c+\nu}{2}\Big)-\varphi'\Big(\frac{\sigma+d}{2}\Big)\Big]\Big|\leqslant \leqslant\frac{h^4\,(\nu-\sigma)^4}{1152}\left[|\varphi'''(\nu)|+|\varphi'''(\sigma)|\right]\qquad(23)$$

holds. Here $h\in(0,1]$, $c=h\sigma+(1-h)\nu$ *and* $d=(1-h)\sigma+h\nu$.

Proof. Because $|\varphi'''|$ is a convex fuction, then (18) gives, for the (I_1-I_2):

$$|I_1-I_2|\leqslant|I_1|+|I_2|=\int_0^{\frac{h}{2}}\xi^2(h-2\xi)\left[|\varphi'''((1-h+\xi)\,\sigma+(h-\xi)\nu)|+\right. \left.+|\varphi'''((h-\xi)\sigma+(1-h+\xi)\,\nu)|\right]d\xi= =\left[|\varphi'''(\nu)|+|\varphi'''(\sigma)|\right]\int_0^{\frac{h}{2}}\left|\xi^2(h-2\xi)\right|d\xi=\frac{h^4}{96}\left[|\varphi'''(\nu)|+|\varphi'''(\sigma)|\right].$$

Multiplying both sides of the last inequality by the expression $\dfrac{(\nu-\sigma)^4}{12}$, *we get (23). Proof is completed.* □

Remark. *In (23), if we take* $h = 1$, *we have:*

$$\Bigg|\int_{\sigma}^{\nu}\varphi(\xi)d\xi-\frac{\nu-\sigma}{6}\Big[\varphi\left(\sigma\right)+\varphi\left(\nu\right)+4\varphi\Big(\frac{\sigma+\nu}{2}\Big)\Big]\Bigg|\leqslant$$
$$\leqslant\frac{(\nu-\sigma)^4}{1152}\left[|\varphi'''(\nu)|+|\varphi'''(\sigma)|\right]. \quad (24)$$

Remark. *This estimate was obtained by Hussain and Qaisar in [14] (Theorem 2) and Bayraktar in [4] (see Remark 4.1)*

Remark. *In (23), if we take* $h = \frac{1}{2}$, *we obtain*

$$\Bigg|\int_{\frac{3\sigma+\nu}{4}}^{\frac{\sigma+3\nu}{4}}\varphi(\xi)d\xi-\frac{\nu-\sigma}{6}\Big[\varphi\Big(\frac{3\sigma+\nu}{4}\Big)+\varphi\Big(\frac{\sigma+\nu}{2}\Big)+\varphi\Big(\frac{\sigma+3\nu}{4}\Big)\Big]\Bigg|+$$
$$+\Bigg|\frac{(\nu-\sigma)^2}{96}\Big[\varphi'\Big(\frac{\sigma+3\nu}{2}\Big)-\varphi'\Big(\frac{3\sigma+\nu}{2}\Big)\Big]\Bigg|\leqslant$$
$$\leqslant\frac{(\nu-\sigma)^4}{16\cdot 1152}\left[|\varphi'''(\nu)|+|\varphi'''(\sigma)|\right]. \quad (25)$$

Theorem 6. *Let* $\varphi\colon I\to\mathbb{R}$ *and* $\varphi\in C^3(I^\circ)$. *For* $0\leqslant\sigma<\nu$, *suppose that* $\sigma,\nu\in I^\circ$ *and* φ''' *satisfies the inequality (13) on* $[\sigma,\nu]$. *Then the inequality:*

$$\Bigg|\Bigg(\int_{c}^{\frac{\nu+c}{2}}\varphi(\xi)d\xi+\int_{\frac{\sigma+d}{2}}^{d}\varphi(\xi)d\xi\Bigg)-$$
$$-\frac{h(\nu-\sigma)}{6}\Big[\varphi(d)+\varphi(c)+2\cdot\Big(\varphi\Big(\frac{\sigma+d}{2}\Big)+\varphi\Big(\frac{c+\nu}{2}\Big)\Big)\Big]\Bigg|+$$
$$+\Bigg|\frac{h^2(\nu-\sigma)^2}{24}\Big[\varphi'\Big(\frac{c+\nu}{2}\Big)-\varphi'\Big(\frac{\sigma+d}{2}\Big)\Big]\Bigg|\leqslant$$
$$\leqslant\frac{h^4\left(|7h-5|\right)(\nu-\sigma)^4}{5760}\left|\varphi'''(\nu)-\varphi'''(\sigma)\right| \quad (26)$$

holds. Here, $h \in (0,1]$, $c = h\sigma + (1-h)\nu$ *and* $d = (1-h)\sigma + h\nu$.

Proof. *For the* $(I_1 - I_2)$ *from (18), by taking into account (13), we get*

$$|I_1 - I_2| \leqslant \int_0^{\frac{h}{2}} \xi^2(h-2\xi)\,|\varphi'''((1-h+\xi)\,\sigma + (h-\xi)\nu) -$$
$$-\varphi'''((h-\xi)\sigma + (1-h+\xi)\,\nu)|\,d\xi =$$
$$= |\varphi'''(\nu) - \varphi'''(\sigma)| \int_0^{\frac{h}{2}} \xi^2(h-2\xi)\,|2h-2\xi-1|\,d\xi.$$

Since $\forall \xi \in \left[0, \frac{h}{2}\right]$, $\begin{cases} 2h-2\xi-1 < 0, & h \in [0;0{,}5) \\ 2h-2\xi-1 \geqslant 0, & h \in [0{,}5;1] \end{cases}$,

$$|I_1 - I_2| = |\varphi'''(\nu) - \varphi'''(\sigma)| \left| \int_0^{\frac{h}{2}} \xi^2(h-2\xi)\,(2\xi-2h+1)\,d\xi \right| =$$
$$= \frac{h^4\,|7h-5|}{16 \cdot 30}\,|\varphi'''(\nu) - \varphi'''(\sigma)|\,.$$

Multiplying both sides of the last inequality by the expression $\dfrac{(\nu-\sigma)^4}{12}$, *we get (26). Proof is completed.* □

Remark. *In (26), if we take* $h = 1$, *we obtain*

$$\left| \int_\sigma^\nu \varphi(\xi)d\xi - \frac{\nu-\sigma}{6}\left[\varphi(\sigma) + \varphi(\nu) + 4\varphi\Big(\frac{\sigma+\nu}{2}\Big)\right] \right| \leqslant$$
$$\leqslant \frac{(\nu-\sigma)^4}{2880}\,|\varphi'''(\nu) - \varphi'''(\sigma)| \quad (27)$$

Proposition 1. *Estimate of the upper bound (27) for Simpson's inequality is better than estimate (2).*

Proof. Indeed, from Lagrange's theorem, for the right-hand side of (27), we have:

$$\frac{(\nu-\sigma)^4}{2880}|\varphi'''(\nu)-\varphi'''(\sigma)| = \frac{(\nu-\sigma)^4}{2880}|\varphi^{(4)}(\xi)|(\nu-\sigma) \leqslant \frac{(\nu-\sigma)^5}{2880}\left\|\varphi^{(4)}\right\|_\infty, \quad (28)$$

where $\xi \in (\sigma, \nu)$. □

Remark. *In (23), if we take* $h = \dfrac{1}{2}$, *we have*

$$\left| \int\limits_{\frac{3\sigma+\nu}{4}}^{\frac{\sigma+3\nu}{4}} \varphi(\xi)d\xi - \frac{\nu-\sigma}{6}\left[\varphi\Big(\frac{3\sigma+\nu}{4}\Big) + \varphi\Big(\frac{\sigma+\nu}{2}\Big) + \varphi\Big(\frac{\sigma+3\nu}{4}\Big)\right]\right| + \\ + \left| \frac{(\nu-\sigma)^2}{96}\left[\varphi'\Big(\frac{\sigma+3\nu}{2}\Big) - \varphi'\Big(\frac{3\sigma+\nu}{2}\Big)\right]\right| \leqslant \\ \leqslant \frac{(\nu-\sigma)^4}{16\cdot 3840}\left|\varphi'''(\nu) - \varphi'''(\sigma)\right|. \quad (29)$$

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