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ANALYTIC FUNCTIONS OF INFINITE ORDER IN HALF-PLANE

Abstract. J. B. Meles (1979) considered entire functions with zeros restricted to a finite number of rays. In particular, it was proved that if f is an entire function of infinite order with zeros restricted to a finite number of rays, then its lower order equals infinity. In this paper, we prove a similar result for a class of functions analytic in the upper half-plane. The analytic function f in $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ is called proper analytic if $\limsup\limits_{z\to t} \ln|f(z)| \leqslant 0$ for all real numbers $t \in \mathbb{R}$. The class of the proper analytic functions is denoted by JA. The full measure of a function $f \in JA$ is a positive measure, which justifies the term "proper analytic function". In this paper, we prove that if a function f is the proper analytic function in the half-plane $\mathbb{C}_+$ of infinite order with zeros restricted to a finite number of rays $\mathbb{L}_k$ through the origin, then its lower order equals infinity.

Key words: *half-plane, proper analytic function, infinite order, lower order, Fourier coefficients, full measure*

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1. Introduction, Definitions and Notations. In the 60s of the last century, several authors (L. A. Rubel and B. A. Taylor [16], J. B. Miles [11], J. B. Miles and D. F. Shea [13], [14], and others) started to use the Fourier series method on a large scale for studying of the properties of entire and meromorphic functions. This method is efficient in solution of several general problems of the theory of meromorphic functions and establishes its connections with Fourier series theory. One advantage of this method is its suitability for investigation of functions of fairly irregular growth at infinity and functions of infinite order. In the 80s of the last century, important results in this direction were obtained by A. A. Kondratyuk [5], [6], [7], who generalized the Levin–Pflüger theory of entire functions of completely regular growth to meromorphic functions of the gamma-type. At the beginning of this century, the first author of this paper extended the method of Fourier series to delta-subharmonic functions in the half-plane [10].

For meromorphic functions $f(z)$ in complex plane $\mathbb{C}$, the characteristic functions are defined as

$$m(r,f) = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln^+ |f(re^{i\varphi})|\, d\varphi\,,$$

$$N(r,f) = \int\limits_0^{r} \frac{n(t,f)-n(0,f)}{t}\, dt + n(0,f)\ln r\,,$$

where $n(r,f)$ is the number of poles of f with regard to their multiplicity in the disk $B(0,r)$ ($C(a,r)=\{z : |a-z|<r\}$, $B(a,r)=\overline{C(a,r)}$, $\overline{G}$ means the closure of a set G), $x^+ = \max\{0; x\}$,

$$T(r,f) = m(r,f) + N(r,f)\,.$$

The order and the lower order of a meromorphic function f are, respectively, the values

$$\beta[f] = \limsup_{r\to\infty} \frac{\ln\ T(r,f)}{\ln r}, \quad \alpha[f] = \liminf_{r\to\infty} \frac{\ln T(r,f)}{\ln r}\,.$$

The order and the lower order of the entire function f are, respectively, the values

$$\beta[f] = \limsup_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r}, \quad \alpha[f] = \liminf_{r\to\infty} \frac{\ln\ln M(r,f)}{\ln r}\,,$$

where $M(r,f) = \max\limits_{0\leqslant\theta\leqslant 2\pi} |f(re^{i\theta})|$.

In the work [12], J. B. Meles considered entire functions with zeros restricted to a finite number of rays. In particular, it was proved that if f is an entire function of infinite order with zeros restricted to a finite number of rays, then its lower order equals infinity.

We prove a similar result for functions that are analytic in the half-plane. To prove this statement, we use the Fourier series method developed by the first author [10]. Passing to the half-plane is complicated by complex behaviour of the function near the boundary. The difference from the plane is obvious just in obtaining tests for a meromorphic function to belong to a fixed class (see [9, Teorems 1, 2, and the Example after Theorem 2]).

A special case, when zeros are restricted to the imaginary axis, was considered by the third author [17]. Note that the case of zeros at an arbitrary system of rays is more complicated than one ray. In particular, the proof in this case essentially uses Lemma 2.

The main result is the following theorem.

Theorem 1. *Suppose f is the proper analytic function in the half-plane $\mathbb{C}_+$ of infinite order with zeros restricted to a finite number of rays $\mathbb{L}_k$ through the origin:*

$$\mathbb{L}_k = \left\{ z : \arg z = e^{i\theta_k}, \quad 0 < \theta_k < \pi, k \in \overline{1, N_0}, N_0 \in \mathbb{N} \right\} .$$

Then its lower order equals infinity.

2. Classes of Functions in $\mathbb{C}_+$. Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ be the upper half-plane. We denote by Ω_+ the intersection of a set Ω with the half-plane $\mathbb{C}_+ : \Omega_+ = \Omega \cap \mathbb{C}_+$. If $0 < r_1 < r_2$, then $D_+(r_1, r_2) = \overline{C_+(0, r_2) \setminus C_+(0, r_1)}$ means the closed half-ring.

Let AK be the class of analytic functions $f(z)$ in $\mathbb{C}_+$, such that $\ln|f(z)|$ possess a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. Functions $f(z)$ in AK have the following properties [1]:

a) $\ln|f(z)|$ has non-tangential limits $\ln|f(t)|$ almost everywhere on the real axis and $\ln|f(t)| \in L^1_{\text{loc}}(-\infty, \infty)$;

b) there exists a measure (charge) of variable sign ν on the real axis, such that

$$\lim_{y \to +0} \int_a^b \ln|f(t+iy)|\, dt = \nu([a,b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\}) .$$

The measure ν is called *the boundary measure* of f;

c) $d\nu(t) = \ln|f(t)|\, dt + d\sigma(t)$, where σ is a singular measure with respect to the Lebesgue measure.

For a function $f \in AK$, we define, following [1], *the full measure* $\lambda_f := \lambda$ by the formula

$$\lambda(G) = 2\pi \int_{\mathbb{C}_+ \cap G} \operatorname{Im} \zeta \, d\mu(\zeta) - \nu(G) ,$$

where μ is the Riesz measure of $\ln|f(z)|$.

The full measure determines a function $f \in AK$ in the same way as the Riesz measure μ determines a subharmonic function in $\mathbb{C}$. More precisely, if $f_1, f_2 \in AK$ have the same full measure λ, then there exists a real entire function g, such that $|f_1(z)| - \ln|f_2(z)| = \operatorname{Im} g(z)$, $z \in \mathbb{C}_+$.

A function f is called *proper analytic* in $\mathbb{C}_+$ if f is an analytic function in $\mathbb{C}_+$ and $\limsup\limits_{z\to t} \ln|f(z)| \leqslant 0$ for each $t \in \mathbb{R}$. The full measure of the proper analytic function is a positive measure, which explains the term «proper analytic function». Note that a proper analytic function is necessarily in the class AK. By JA we denote the class of proper analytic functions in $\mathbb{C}_+$.

A function f is called *proper meromorphic function* in $\mathbb{C}_+$ if f can be represented as a quotient of two proper analytic functions. The class of proper meromorphic functions in $\mathbb{C}_+$ is denoted by JM.

Proposition 1. [1] $JM = AK/AK$.

Let $f_1, f_2 \in JA$. The functions f_1 and f_2 *have no common zeros* in $\overline{\mathbb{C}_+}$ if full measures of f_1 and f_2 are mutually singular.

Proposition 2. [1] For $f \in JM$, the identity $f = f_1/f_2$ holds; here f_1, f_2 are proper analytic functions without common zeros in $\overline{\mathbb{C}_+}$. They are defined uniquely up to a multiplier of the form $e^{ig(z)}$, where g is a real entire function.

Note, however, that the uniqueness of the numerator and denominator in Proposition 2 is not so simple to prove. For example, both could be also multiplied by constant less then 1.

3. Nevanlinna's Characteristic Functions for the Complex Half–Plane. In 1925, R. Nevanlinna [15] (see also [3]) introduced for meromorphic function $f(z)$ in the closed upper half-plane $\overline{\mathbb{C}_+}$ the characteristic functions

$$A(r, f) = \frac{1}{\pi} \int\limits_0^r \left(\frac{1}{t^2} - \frac{1}{r^2} \right) \left(\ln^+ |f(t)| + \ln^+ |f(-t)| \right) dt\,,$$

$$B(r, f) = \frac{2}{\pi r} \int\limits_0^\pi \ln^+ |f(re^{i\varphi})| \sin\varphi \, d\varphi\,,$$

$$C(r,f) = 2\int_0^r \left(\frac{1}{t^2} + \frac{1}{r^2}\right) c(t,f)\,dt\,,$$

where $c(r,f) = \sum_{1\leqslant r_n \leqslant r} \sin\varphi_n$, $r_n e^{i\varphi_n}$ are poles of f with regard to their multiplicity,

$$S(r,f) = A(r,f) + B(r,f) + C(r,f)\,.$$

The following question arises: what is a good analogue of the functions $m(r,f)$ and $N(r,f)$ for $\mathbb{C}_+$? R. Nevanlinna [15], A. A. Goldberg [2], A. A. Goldberg and I.V. Ostrovskii [3] reasoned that the analogue of the function $m(r,f)$ is $A(r,f)+B(r,f)$ and the analogue of $N(r,f)$ is $C(r,f)$.

We will exploit other definitions as well to emphasize the connection with the case of the complex plane. These characteristic functions are defined in the widest space where they have meaning. Here we use terminology and definitions from [1].

Let $f \in JM$ and let λ be its full measure. Let $\lambda = \lambda_+ - \lambda_-$ be the Jordan decomposition of the measure λ. We denote

$$m(r,f) := \frac{1}{r}\int_0^\pi \ln^+|f(re^{i\varphi})|\sin\varphi\,d\varphi = \frac{\pi}{2}B(r,f)\,,$$

$$N(r,f,r_0) := \int_{r_0}^r \frac{\lambda_-(t)}{t^3}\,dt = \frac{\pi}{2}(A(r,f) + C(r,f) + O(1))\,,$$

$$T(r,f,r_0) := m(r,f) + N(r,f,r_0) + m(r_0,1/f) = \frac{\pi}{2}S(r,f) + O(1)\,,$$

where r_0 is an arbitrary fixed positive number (one may as well take $r_0 = 1$), which will be omitted in designations provided that this does not cause any misunderstanding; for example, instead of $T(r,f,r_0)$ we will write $T(r,f)$, and so on, $r \geqslant r_0$, $\lambda_-(t) = \lambda_-(B(0,t))$.

Definition 1. *The order and the lower order of the function $f \in JM$ are the values*

$$\beta[f] = \limsup_{r\to\infty} \frac{\ln(rT(r,f))}{\ln r}, \quad \alpha[f] = \liminf_{r\to\infty} \frac{\ln(rT(r,f))}{\ln r}\,.$$

4. Preliminaries. Let λ be the full measure of a function $f \in JM$. The function f has the following representation in the half-disc $C_+(0,R)$ [4]:

$$\ln|f(z)| = -\frac{1}{2\pi}\iint\limits_{\overline{C_+(0,R)}} \frac{G(z,\zeta)}{\operatorname{Im}\zeta}\,d\lambda(\zeta)+ \\ + \frac{R}{2\pi}\int\limits_0^\pi \frac{\partial G(z,Re^{i\varphi})}{\partial\tau}\ln|f(Re^{i\varphi})|\,d\varphi, \quad z\in C_+(0,R)\,, \quad (1)$$

where $G(z,\zeta)$ is the Green function of the half-disc, $\dfrac{\partial G}{\partial\tau}$ is its derivative in the inward-normal direction, and the kernel $\dfrac{G(z,\zeta)}{\operatorname{Im}\zeta}$, where $\zeta\in\overline{C_+(0,R)}$, is extended by continuity to the real axis for $|t|\leqslant R$.

For the measure λ, let

$$d\lambda_k(\tau e^{i\varphi}) = \frac{\sin k\varphi}{\sin\varphi}\tau^{k-1}\,d\lambda(\tau e^{i\varphi}), \quad \lambda_k(r)=\lambda_k(\overline{C(0,r)}), \quad k\in\mathbb{N}\,,$$

where the function $\dfrac{\sin k\varphi}{\sin\varphi}$ is defined for $\varphi=0$ and $\varphi=\pi$ by continuity.

The next relation is Carleman's formula in Grishin's notation [4]:

$$\frac{1}{r^k}\int\limits_0^\pi \ln|f(re^{i\varphi})|\sin k\varphi\,d\varphi = \int\limits_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\,dt+ \\ + \frac{1}{r_0^k}\int\limits_0^\pi \ln|f(r_0e^{i\varphi})|\sin k\varphi\,d\varphi, \quad k\in\mathbb{N}\,.$$

In particular, for $k=1$ we have

$$\frac{1}{r}\int\limits_0^\pi \ln|f(re^{i\varphi})|\sin\varphi\,d\varphi = \int\limits_{r_0}^r \frac{\lambda(t)}{t^3}\,dt + \frac{1}{r_0}\int\limits_0^\pi \ln|f(r_0e^{i\varphi})|\sin\varphi\,d\varphi\,. \quad (2)$$

In this notation, Carleman's formula (2) can be written as follows:

$$T(r,f) = T(r,1/f)\,. \quad (3)$$

The Fourier coefficients of a function $f\in JM$ are defined by the formula [10]:

$$c_k(r,f) = \frac{2}{\pi}\int\limits_0^\pi \ln|f(re^{i\varphi})|\sin k\theta\,d\theta, \quad k\in\mathbb{N}\,.$$

Let λ be the full measure of $f \in JM$; then [10]:

$$c_k(r,f) = \alpha_k r^k + \frac{2r^k}{\pi} \int\limits_{r_0}^{r} \frac{\lambda_k(t)}{t^{2k+1}}\,dt, \quad k \in \mathbb{N}\,, \tag{4}$$

where $\alpha_k = r_0^{-k} c_k(r_0, f)$, and from (4) we obtain:

$$\begin{aligned} c_k(r,f) = \alpha_k r^k + \frac{r^k}{\pi k r_0^{2k}} \iint\limits_{C_+(0,r_0)} \frac{\sin k\varphi}{\operatorname{Im}\zeta} \tau^k\, d\lambda(\zeta) + \\ + \frac{r^k}{\pi k} \iint\limits_{D_+(r_0,r)} \frac{\sin k\varphi}{\tau^k \operatorname{Im}\zeta}\, d\lambda(\zeta) - \frac{1}{r^k \pi k} \iint\limits_{C_+(0,r)} \frac{\sin k\varphi}{\operatorname{Im}\zeta} \tau^k\, d\lambda(\zeta)\,, \end{aligned} \tag{5}$$

where $\zeta = \tau e^{i\varphi}$.

From definition of $c_k(r,f)$, the inequality follows:

$$|c_k(r,f)| \leqslant \frac{2k}{\pi} \int\limits_0^{\pi} |\ln|f(re^{i\varphi})||\sin\varphi\, d\varphi, \quad k \in \mathbb{N}\,.$$

From this and from equality (3), we obtain

$$rm(r,f) \geqslant \frac{\pi}{4k} |c_k(r,f)|, \quad k \in \mathbb{N}\,. \tag{6}$$

Indeed, (6) follows from the relations

$$\begin{aligned} \frac{\pi}{2kr}|c_k(r,f)| \leqslant \frac{1}{r} \int\limits_0^{\pi} (\ln^+|f(re^{i\varphi})| + \ln^+|1/f(re^{i\varphi})|)\sin\varphi\, d\varphi \leqslant \\ \leqslant m(r,f) + m(r,1/f) \leqslant 2m(r,f), \quad k \in \mathbb{N}\,. \end{aligned}$$

Now we need a lemma:

Lemma 1. *If $g \in JM$ and $\lambda_g \equiv 0$, then $\ln|g(z)| = \operatorname{Im} F(z)$, where $F(z)$ is an entire real function.*

Proof. Remind [8] that the entire function $F(z)$ is a real function if $F(\mathbb{R}) \subset \mathbb{R}$.

Since the full measure of the function g equals zero, then from (1) it follows that for any $R > 0$

$$\ln|g(z)| = \frac{R}{2\pi}\int_0^{\pi} \frac{\partial G(z, Re^{i\varphi})}{\partial n} \ln|g(Re^{i\varphi})|\,d\varphi\,, \quad z \in C_+(0,R)\,.$$

The right-hand part is a harmonic function in a half-disc $C_+(0,R)$, which is extended continuously by zero to the interval $(-R,R)$. As R is an arbitrary positive number, then the function $\ln|g(z)|$ is harmonic in the half-plane $\mathbb{C}_+$ and is extended continuously by zero to the real axis. By the principle of symmetry, this function is extended as harmonic on the bottom half-plane.

Then there exists a harmonic function $h(z)$ on the complex plane vanishing on the real axis and such that $\ln|g(z)| = h(z)$ for $\operatorname{Im} z > 0$.

Let $(-h_1(z))$ be a function harmoniously conjugated to the function $h(z)$. Then $F(z) = h_1(z) + ih(z)$ is an entire function, real on the real axis and $\ln|g(z)| = \operatorname{Im} F(z)$. □

The following lemma [12, Lemma 1.1] is used in the proof of Theorem 1.

Lemma 2. *Suppose $\theta_1, \theta_2, \ldots, \theta_{N_0}$ are distinct elements of $[0, 2\pi)$. For a real x, let x^* denote the unique number in $[-\pi, \pi)$, congruent to x modulo 2π. There exists an increasing sequence $I = \{n_l\}$ of positive integers, such that I has positive density and*

$$(n_l\theta_j)^* \in \Big(-\frac{\pi}{6}, \frac{\pi}{6}\Big) \tag{7}$$

for $j \in \overline{1, N_0}$ and $n_l \in I$.

3. Proof of the Theorem 1. Since λ_f restricted to the finite system of rays, then, by formula (5), we obtain for the Fourier coefficients of the function f:

$$c_n(r,f) = \alpha_n r^n + \sum_{k=1}^{N_0} \frac{r^n \sin(\theta_k n)}{\pi n r_0^{2n} \sin\theta_k} \int_0^{r_0} t^{n-1} d\lambda(t) +$$
$$+ \sum_{k=1}^{N_0} \frac{r^n \sin(\theta_k n)}{\pi n \sin\theta_k} \int_{r_0}^{r} \frac{d\lambda(t)}{t^{n+1}} - \sum_{k=1}^{N_0} \frac{\sin(\theta_k n)}{r^n \pi n \sin\theta_k} \int_0^{r} t^{n-1} d\lambda(t),\ n \in \mathbb{N}\,.$$

Then we obtain

$$c_n(r,f)=\alpha_n r^n+\sum_{k=1}^{N_0}\frac{r^n\sin(\theta_k n)}{\pi n r_0^{2n}\sin\theta_k}\int\limits_0^{r_0}t^{n-1}d\lambda(t)+$$
$$+\sum_{k=1}^{N_0}\frac{\sin(\theta_k n)}{\pi n\sin\theta_k}\int\limits_{r_0}^{r}\frac{1}{t}\left[\left(\frac{r}{t}\right)^n-\left(\frac{t}{r}\right)^n\right]d\lambda(t),\, n\in\mathbb{N}\,. \quad (8)$$

Integrating by parts twice in (8), we get

$$c_n(r,f)=\alpha_n r^n+\sum_{k=1}^{N_0}\frac{r^n\sin(\theta_k n)}{\pi n r_0^{2n}\sin\theta_k}\int\limits_0^{r_0}t^{n-1}d\lambda(t)+$$
$$+\frac{2}{\pi}\sum_{k=1}^{N_0}\frac{\sin(\theta_k n)}{\sin\theta_k}\left(\tilde{N}(r)+r^n\int\limits_{r_0}^{r}\frac{\tilde{N}(t)}{t^{n+1}}\,dt\right)+$$
$$+\frac{n-1}{\pi}\sum_{k=1}^{N_0}\frac{\sin(\theta_k n)}{\sin\theta_k}\int\limits_{r_0}^{r}\frac{1}{t}\left[\left(\frac{r}{t}\right)^n-\left(\frac{t}{r}\right)^n\right]\tilde{N}(t)\,dt,\, n\in\mathbb{N}\,, \quad (9)$$

where $\tilde{N}(r)=\int\limits_{r_0}^{r}\frac{\lambda(t)}{t^2}dt.$

By Lemma 2, choose the sequence $I=\{n_l\}$ such that $(n_l\theta_j)^*\in\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$. Then $\sum_{k=1}^{N_0}\sin(\theta_k n_l)=\sum_{k=1}^{N_0}\sin(\theta_k n_l)^*\geqslant N_0\sin\frac{\pi}{6}=\frac{N_0}{2}$. Note that

$$\sum_{k=1}^{N_0}\frac{r^{n_l}\sin(\theta_k n_l)}{\pi n_l r_0^{2n_l}\sin\theta_k}\int\limits_0^{r_0}t^{n_l-1}d\lambda(t)>0,\, n_l\in\mathbb{N}\,.$$

From (9) with $n=n_l$, $l\in\mathbb{N}$, we obtain

$$\frac{|c_{n_l}(r,f)|}{r^{n_l}}\geqslant\frac{N_0}{\pi}\left(\frac{\tilde{N}(r)}{r^{n_l}}+\int\limits_{r_0}^{r}\frac{\tilde{N}(t)}{t^{n_l+1}}\,dt\right)-|\alpha_{n_l}|,\, n_l\in\mathbb{N}\,. \quad (10)$$

If the function $\tilde{N}(r)$ has infinite order, then the integral in the right-hand part of the last inequality is unlimited as $r \to \infty$, because

$$\int_r^\infty \frac{\tilde{N}(t)}{t^{n+1}}\,dt \geqslant \frac{\tilde{N}(r)}{nr^n}, \quad n \in \mathbb{N},$$

and right-hand part of this inequality can be made arbitrarily large by suitable choice of r. By this inequality, inequality (6), and (10), we receive the demanded statement.

If $\tilde{N}(r)$ has a finite order, then there exist positive numbers $K > 0$ and $\rho > 0$, such that $\tilde{N}(r) \leqslant Kr^\rho$ for all $r > 0$. It is possible to consider non-integer ρ. From here it follows that

$$K2^\rho r^\rho \geqslant \tilde{N}(2r) \geqslant \int_r^{2r} \frac{\lambda(t)}{t^2}\,dt \geqslant \lambda(r)\int_r^{2r} \frac{dt}{t^2} = \frac{\lambda(r)}{2r},$$

i.e.,

$$\lambda(r) \leqslant K2^{\rho+1}r^{\rho+1}.$$

In this case, from the paper [10, Theorem 3] it follows that there exists the function $g_1 \in JA$ of order ρ and with full measure λ. Then the function $g = f/g_1 \in JA$ and $\lambda_g \equiv 0$.

According to the Lemma 1, $|g(z)| = \exp(\operatorname{Im} F(z))$, where $F(z)$ is an entire real function

$$F(z) = \sum_{n=0}^\infty a_n z^n.$$

The fact that $a_n \in \mathbb{R}$, for all $n \in \mathbb{N}$, is proved by termwise derivation of the Taylor series of the function $F(z)$ at the point $z = 0$.

If the only finite number $a_n \neq 0$, then $F(z)$ is a polynomial, hence g and f have a finite order: this contradicts the condition.

Further,

$$c_k(r, g) = a_k r^k, \quad k \in \mathbb{N}.$$

From this and (6), we obtain

$$rm(r, g) \geqslant \frac{\pi}{4k}|c_k(r, g)| = \frac{\pi|a_k|r^k}{4k}.$$

The last inequality is valid for any fixed natural k. It follows that the function $g(z)$ has infinite lower order.

Further, from the elementary inequality $(\ln(ab))_+ \geqslant (\ln a)_+ - (\ln b)_+$, which is valid for $b > 1$, we obtain

$$m(r, g) \leqslant m(r, f) + m(r, 1/g_1) < m(r, f) + T(r, g_1)\,.$$

Then, from the inequality

$$\liminf_{r\to\infty} rm(r, g) \leqslant \liminf_{r\to\infty} rm(r, f) + \limsup_{r\to\infty} rT(r, g_1)$$

and the relations

$$\liminf_{r\to\infty} rm(r, g) = \infty, \quad \limsup_{r\to\infty} rT(r, g_1) \leqslant \rho < \infty\,,$$

it follows that

$$\liminf_{r\to\infty} rm(r, f) = \infty\,.$$

The theorem 1 is proved. □

Remark. *In Theorem 1 the boundary measure ν of f, generally speaking, is not zero. It is a non-negative measure.*

The boundary measure of a function $f \in AK$ can be sign-variable, its full measure can also be sign-variable. So, I think it has merely been observed that the proof of Theorem 1 does not apply to functions in the class AK. The question whether there exists a specific f in AK of infinite order and finite lower order i still open.

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