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ON MULTIVALUED $\perp_{\psi F}$ - CONTRACTIONS ON GENERALIZED ORTHOGONAL SETS WITH AN APPLICATION TO INTEGRAL INCLUSIONS

Abstract. We study existence of fixed points for multivalued $\perp_{\psi F}$-contractions in the setting of generalized orthogonal sets by extending some basic notions related to this new direction of research. The proven theorems generalize and improve many known results in the literature. Also, an application to a Volterra-type integral inclusion is provided.

Key words: *multivalued $\perp_{\psi F}$-Contractions,Fixed point, generalized orthogonal set, generalized orthogonal complete metric space, integral inclusion*

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1. Introduction. Nadler [6] (1969) was the first author who combined the notion of Hausdorff metric and contractions and proved a fixed-point theorem for this class of contractions. Since then, this type has been dealt with in a number of papers [2], [9]. In 2015, Altun et al [1] introduced multivalued F-contractions by using the idea of Wardowski [14] (2012) and Nadler [6]. Also, a fixed-point result for this class of mappings was proven. On the other hand, Gordji et al [4] (2014) defined the notion of orthogonal set, and, hence, a generalization of the Banach contraction. After that, Baghani et al [3] (2017) gave a generalization of F-contraction on orthogonal sets called $\perp_F$-contraction and established a fixed-point result for these contractions. Other works in this area can be found in [11, 13].

Very recently, the authors in [10] (2020) have introduced the notion of generalized orthogonal sets and some related basic concepts as an extension of orthogonal sets. Further, they proved some fixed-point theorems for $\perp_{\psi F}$-contraction mappings.

In this paper, motivated by the major role of fixed points for multi-valued mappings, we generalize the notion of $\perp_{\psi F}$-contractions to mutli-valued $\perp_{\psi F}$-contractions. Also, we extend some related notions and prove new fixed-point theorems for this new direction of research. In this work, we show the superiority of the obtained results compared to the existing ones in the literature ([3], [4], [10]). Finally, as an extension of some applications from the literature [10], [12], an application to a Volterra-type integral inclusion under new weak conditions is considered.

2. Preliminary. Throughout this article, (X, d) is a metric space and $CB(X)$ (respectively, $K(X)$) denotes the family of all nonempty closed and bounded subsets of X (respectively, of compact subsets of X). Define

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

for a given $A, B \in CB(X)$ with $d(a, B) = \inf\{d(a, b) : b \in B\}$. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d. Now, we describe some notions and results used in the sequel.

Definition 1. *[8], [14] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping and consider the following conditions:*

(F1) F is strictly increasing;

(F2) For each sequence $\{\alpha_n\}$ of positive numbers, we get

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) There exists $\lambda \in (0, 1)$, such that $\lim_{\alpha\to 0} \alpha^\lambda F(\alpha) = 0$.

$\mathcal{F}$ denotes the class of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy conditions (F1), (F2), and (F3).

Definition 2. *[14] A mapping $T : X \to X$ is said to be an F-contraction, where $F \in \mathcal{F}$, if*

$$\exists\tau > 0, \forall x, y \in X, d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)).$$

Definition 3. *[1] A mapping $T : X \to CB(X)$ is said to be an F-contraction, where $F \in \mathcal{F}$, if*

$$\exists\tau > 0, \forall x, y \in X, H(Tx, Ty) > 0 \implies \tau + F(H(Tx, Ty)) \leqslant F(d(x, y)).$$

Theorem 1. *[1] Let (X, d) be a complete metric space and $T\colon X \to K(X)$ be a mutlivalued F-contraction; then T has a fixed point in X.*

Definition 4. *[8] Let Ψ denote the family of all functions $\psi\colon \mathbb{R} \to \mathbb{R}$ that satisfy the following assumptions:*

(ψ1) ψ is increasing;

(ψ2) $\psi^n(t) \to -\infty$ for every $t \in \mathbb{R}$.

Lemma 1. *[8] If $\psi \in \Psi$, then $\psi(t) < t$ for all $t \in \mathbb{R}$.*

Definition 5. *[8] A mapping $T\colon X \to X$ is said to be an ψF-contraction, where $F \in \mathcal{F}$ and $\psi \in \Psi$, if*

$$\forall x, y \in X, d(Tx, Ty) > 0 \implies F(d(Tx, Ty)) \leqslant \psi[F(d(x, y))].$$

Remark 1. *[8] If we take in Definition 5 $\psi(t) = t - \tau$, $\tau > 0$, we get the F-contraction in Definition 2.*

Lemma 2. *[7, Lemma 2.2] Let (X, d) be a metric space and $A, B \in CB(X)$. If there exists $\gamma > 0$, such that:*

i) For each $a \in A$, there is a $b \in B$, so that $d(a, b) \leqslant \gamma$;

ii) For each $b \in B$, there is an $a \in A$, so that $d(b, a) \leqslant \gamma$,

then $H(A, B) \leqslant \gamma$.

Now, we recall the definition of orthogonal sets, generalized orthogonal sets, and some related basic concepts.

Definition 6. *[4] Let $X \neq \emptyset$ and let $\perp \subset X \times X$ be a binary relation. If $\perp$ satisfies the following assumption:*

$$\exists x_0 : (\forall y, y \perp x_0) \text{ or } (\forall y, x_0 \perp y), \tag{1}$$

then it is called an orthogonal set.

Definition 7. *[10] Let $X \neq \emptyset$ and let $\perp_g \subset X \times X$ be a binary relation, such that $\perp_g$ satisfies the following condition:*

$$\exists x_0, \forall y \in X \setminus \{x_0\}, \quad y \perp_g x_0 \quad \text{or} \quad x_0 \perp_g y; \tag{2}$$

then it is called a generalized orthogonal set. We denote it by $(X, \perp_g)$. Also, the element x_0 is said to be a generalized orthogonal element.

Example 1. [10] Let $X = \mathbb{R}$. Define a binary relation $\perp_g$ on X by

$$x \perp_g y \Longleftrightarrow x < y. \tag{3}$$

It is easy to see that $(X, \perp_g)$ is a generalized orthogonal set, but not an orthogonal set.

Remark 2. *As noted in [10], the generalized orthogonal element is not unique. In the above example, one can see that every element $x \in X$ is a generalized orthogonal element.*

Example 2. [10] Let (X, τ) be a topological space. We define a binary relation $\perp_g$ on $X \times X$ by

$$A \perp_g B \Longleftrightarrow \overline{A} \subseteq \overset{\circ}{B} \text{ and } A \neq B;$$

$(X, \perp_g)$ is a generalized orthogonal set, but not an orthogonal set (the converse is not true) and $\emptyset$ is a generalized orthogonal element.

Definition 8. *[10] Let $(X, \perp_g)$ be a generalized orthogonal set. A sequence $\{x_n\} \subset X$ is called a generalized orthogonal sequence, if for all $n \in \mathbb{N}$,*

$$x_n \neq x_{n+1} \implies x_n \perp_g x_{n+1} \text{ or } x_{n+1} \perp_g x_n.$$

Definition 9. *[10] The triplet $(X, \perp_g, d)$ is said to be a generalized orthogonal metric space, if (X, d) is a metric space and $(X, \perp_g)$ is a generalized orthogonal set.*

Definition 10. *[10] Let $(X, \perp_g, d)$ be a generalized orthogonal metric space and $T\colon X \to X$ be a self-mapping. T is said to be generalized $\perp_g$ preserving, if for all $x, y \in X$,*

$$x \perp_g y \text{ and } d(Tx, Ty) > 0 \implies Tx \perp_g Ty.$$

Definition 11. *[10] Let $(X, \perp_g, d)$ be a generalized orthogonal metric space. X is called a generalized orthogonal complete space, if every Cauchy generalized orthogonal sequence $\{x_n\} \subset X$ is convergent.*

3. Main results.

In this section, we start with the following definition:

Definition 12. *Let $(X, \perp_g, d)$ be a generalized orthogonal metric space and $T : X \to CB(X)$ be a mutlivalued mapping. T is said to be multivalued generalized $\perp_g$-preserving, if for all $x, y \in X$:*

$$x \perp_g y \text{ and } H(Tx, Ty) > 0 \implies a \perp_g b$$

for all $a \in Tx$ and $b \in Ty$, such that $a \neq b$ (in this case we denote it $Tx \perp_g Ty$).

Example 3. Let $X = \{1, 2, 3\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$ be the usual metric on X. Define a binary relation on X by

$$x \perp_g y \iff x < y, xy \in \{x, y\}.$$

Therefore, $(X, \perp_g, d)$ is a generalized orthogonal metric space and 1 is an orthogonal element. Consider the multivalued mapping $T\colon X \to K(X)$ defined by

$$T(x) = \begin{cases} \{1, 2\}, & \text{if } x = 1, 3; \\ \{2\}, & \text{if } x = 2. \end{cases}$$

Hence, T is a multivalued generalized orthogonal preserving. Indeed, let $x, y \in X$; then $x \perp_g y$ and $H(Tx, Ty) > 0$ imply $x = 1$ and $y = 2$ and, hence, for all $a \in Tx$ and $b \in Ty$, such that $a \neq b$, we have $a \perp_g b$.

Definition 13. *Let $(X, \perp_g, d)$ be a generalized orthogonal metric space; a mapping $T : X \to X$ is called multivalued generalized orthogonal continuous at $x \in X$ if for any generalized orthogonal sequence $\{x_n\} \subset X$ we have*

$$x_n \to x \text{ with respect to } d \implies Tx_n \to Tx \text{ with respect to } H.$$

Example 4. Under the same assumption as in the above example, we define a multivalued mapping $T : X \to K(X)$ by

$$T(x) = \begin{cases} \{1\}, & \text{if } x = 1, 2; \\ \{2, 3\}, & \text{if } x = 3. \end{cases}$$

Then it is clear that T is multivalued generalized orthogonal continuous.

Now, we introduce the notion of generalized orthogonal multivalued ψF-contraction and show some fixed-point theorems for this type of generalized orthogonal metric spaces.

Definition 14. *Let $(X, \perp_g, d)$ be a generalized orthogonal metric space, such that x_0 is a generalized orthogonal element, $F \in \mathcal{F}$, and $\psi \in \Psi$. A multivalued mapping $T\colon X \to CB(X)$ is said to be generalized orthogonal multivalued ψF-contraction (multivalued $\perp_{\psi F}$-contraction) if for all $x, y \in X$:*

$$x \perp_g y \text{ and } H(Tx, Ty) > 0 \implies F(H(Tx, Ty)) \leqslant \psi(F(M(x, y))), \quad (4)$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\,[d(x,Ty) + d(y,Tx)]\}$$

and

$$x_0 \notin Tx_0 \implies \sum_n |\psi^n(D_0)|^{-1/\lambda} \text{ is convergent,} \tag{5}$$

where $D_0 = F(D(x_0, Tx_0)) = F\Big(\sup_{x\in Tx_0} d(x_0,x)\Big)$ *and* $\lambda \in (0,1)$ *is the constant from (F3) in Definition 1.*

The following is the first theorem:

Theorem 2. *Let* $T\colon X \to K(X)$ *be a multivalued mapping on a generalized orthogonal metric space* $(X, \perp_g, d)$*, such that*

i) T *is a multivalued* $\perp_{\psi F}$*-contraction;*

ii) T *is multivalued generalized* $\perp_g$*-preserving;*

iii) T *is multivalued generalized* $\perp_g$*-continuous;*

iv) X *is a generalized orthogonal complete space.*

Then T *has a fixed point in* X*.*

Proof. X is a generalized orthogonal metric space; so there exists an $x_0 \in X$, such that for all $x_0 \neq y \in X$:

$$x_0 \perp_g y \text{ or } y \perp_g x_0. \tag{6}$$

Since Tx_0 is nonempty, we can choose $x_1 \in Tx_0$, if $x_0 = x_1$ or $H(Tx_0, Tx_1) = 0$, so the proof is finished. Otherwise, we obtain $x_0 \perp_g x_1$ or $x_1 \perp_g x_0$ and $H(Tx_0, Tx_1) > 0$. On the other hand, since Tx_1 is closed, we obtain $d(x_1, Tx_1) > 0$ (otherwise $x_1 \in Tx_1$), which implies, by (F1) and (i), that

$$\begin{aligned} F(d(x_1,Tx_1)) &\leqslant F(H(Tx_0,Tx_1)) \leqslant \psi[F(M(x_0,x_1))] \leqslant \\ &\leqslant \psi\Big[F\Big(\max\Big\{d(x_0,x_1), d(x_0,Tx_0), d(x_1,Tx_1), \frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2}\Big\}\Big)\Big] \leqslant \\ &\leqslant \psi[F(\max\{d(x_0,x_1), \frac{1}{2}d(x_0,Tx_1)\})] \leqslant \\ &\leqslant \psi[F(\max\{d(x_0,x_1), \frac{1}{2}[d(x_0,x_1) + d(x_1,Tx_1)]\})] \leqslant \end{aligned}$$

$$\leqslant \psi[F(\max\{d(x_0,x_1),d(x_1,Tx_1)\})] \leqslant \psi[F(d(x_0,x_1))]. \quad (7)$$

From (ii) we get

$$Tx_0 \perp_g Tx_1 \text{ or } Tx_1 \perp_g Tx_0. \quad (8)$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$, such that $d(x_1,x_2)=d(x_1,Tx_1)$. If $x_1 = x_2$, the proof is finished. We suppose that $x_1 \neq x_2$ and, using (8), we obtain $x_1 \perp_g x_2$ or $x_2 \perp_g x_1$. We can suppose that $H(Tx_1,Tx_2) > 0$, which implies, by (i):

$$F(d(x_1,x_2)) \leqslant F(H(Tx_0,Tx_1)) \leqslant \psi[F(d(x_0,x_1))]. \quad (9)$$

By induction, we obtain a sequence $\{x_n\} \subset X$, such that $x_n \neq x_{n+1}$, $x_{n+1} \in Tx_n$, $H(Tx_n,Tx_{n+1}) > 0$, and $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ with:

$$F(d_n) = F(d(x_n,x_{n+1})) \leqslant \psi[F(d(x_{n-1},x_n))] \leqslant \ldots \leqslant \\ \leqslant \psi^n[F(d(x_0,x_1))] \leqslant \psi^n[F(D_0)], \quad (10)$$

for all $n \in \mathbb{N} \cup \{0\}$. By ($\psi$2) and (F2), we obtain

$$\lim_{n\to\infty} d_n = 0. \quad (11)$$

By (F3), we have

$$\lim_{n\to\infty} d_n^\lambda F(d_n) = 0. \quad (12)$$

As $\lim_{n\to\infty} \psi^n[F(D_0)] = -\infty$, there exists $N > 0$, such that $\psi^n[F(D_0)] < 0$ for all $n \geqslant N$ and, hence, by (10), we get

$$d_n^\lambda F(d_n) \leqslant d_n^\lambda \psi^n[F(D_0)] < 0, \forall n \geqslant N. \quad (13)$$

Then, by (12), we have $\lim_{n\to\infty} d_n^\lambda \psi^n[F(d_0)] = 0$. Hence, there exists $N_1 \geqslant N$, such that $d_n^\lambda |\psi^n[F(D_0)]| \leqslant 1$, which implies $d_n \leqslant |\psi^n[F(D_0)]|^{-1/\lambda}$ for all $n \geqslant N_1$. Now, let $p \in \mathbb{N}$ and $n \geqslant N_1$; then we have

$$d(x_n,x_{n+p}) \leqslant d(x_n,x_{n+1}) + \ldots + d(x_{n+p-1},x_{n+p}) = \\ = d_n + \ldots + d_{n+p-1} \leqslant \sum_{k=n}^{n+p-1} |\psi^k[F(D_0)]|^{-1/\lambda}. \quad (14)$$

It follows from $\sum_n |\psi^n[F(d_0)]|^{-1/\lambda} < \infty$, that $\{x_n\}$ is a Cauchy generalized orthogonal sequence. Now, since X is a generalized orthogonal complete metric space, there exists $u \in X$, such that $\lim_{n\to\infty} x_n = u$. On the other hand, we have

$$d(u,Tu) \leqslant d(u,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},Tu) \leqslant \\ \leqslant d(u,x_n) + d(x_n,x_{n+1}) + H(Tx_n,Tu). \quad (15)$$

Finally, since $\{x_n\}$ is a Cauchy generalized orthogonal sequence and T is a multivalued generalized $\perp_g$-continuous, we deduce from (15) that $u \in Tu$. □

Theorem 3. *Let $(X, \perp_g, d)$ be a generalized orthogonal metric space; Theorem 2 holds also if we replace the condition (iii) by the following assumption:*

(iii') If $\{x_n\} \subset X$ is a generalized orthogonal sequence converging to $x \in X$, then $x_n \perp_g x$ or $x \perp_g x_n$ for all $n \in \mathbb{N}$.

Proof. From the proof of Theorem 2 we see that the generalized orthogonal sequence $\{x_n\}$ converges to $u \in X$. Put $\Gamma = \{n \in \mathbb{N} \mid x_{n+1} \in Tu\}$ and consider the following two cases:

Case I: If Γ is an infinite set, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $x_{n(k)+1} \in Tu$ for all $k \in \mathbb{N}$. Since $\{x_n\}$ converges to u, we obtain $u \in Tu$.

Case II: If Γ is a finite set, there exists $N \in \mathbb{N}$, such that $x_{n+1} \notin Tu$ for all $n \geqslant N$, and, hence, $H(Tx_n,Tu) > 0$ for all $n \geqslant N$. On the other hand, as $x_n \perp_g u$ or $u \perp_g x_n$ for all $n \geqslant N$, we obtain, by the fact that T is a mutlivalued $\perp_{\psi F}$-contraction:

$$F(d(x_{n+1},Tu)) \leqslant F(H(Tx_n,Tu)) \leqslant \\ \leqslant \psi\big[F\big(\max\{d(x_n,u), d(x_n,Tx_n), d(u,Tu), \frac{1}{2}[d(x_n,Tu) + d(u,Tx_n)]\}\big)\big] \leqslant \\ \leqslant F\big(\max\{d(x_n,u), d(x_n,x_{n+1}) + d(x_{n+1},Tx_n), \\ d(u,Tu), \frac{1}{2}[d(x_n,Tu) + d(u,Tx_n)]\}\big) \leqslant \\ \leqslant F(\max\{d(x_n,u), d(x_n,x_{n+1}), d(u,Tu), \frac{1}{2}[d(x_n,Tu) + d(u,x_{n+1})]\})$$

for all $n \geqslant N$. If $u \in Tu$, the proof is completed; otherwise, we obtain

$$F(d(x_{n+1},Tu)) \leqslant \\ \leqslant \max\{d(x_n,u), d(x_n,x_{n+1}), d(u,Tu), \frac{1}{2}[d(x_n,Tu) + d(u,x_{n+1})]\}.$$

On the other hand, since $\{x_n\}$ converges to u, we have, due to (F2): $\lim_{n\to\infty} F(d(x_{n+1}, Tu)) = -\infty$; again, by (F2), we conclude that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = 0,$$

which implies $u \in Tu$. □

Corollary. *Let (X, d) be a complete metric space and $T\colon X \to K(X)$ be a multivalued F-contraction. Then T has a fixed point.*

Proof. Define a binary relation on $X \times X$ as follows:

$$x \perp_g y \Leftrightarrow \Big[d(Tx, Ty) > 0 \implies F(d(Tx, Ty)) \leqslant \psi[F(d(x, y))] \Big], \quad (16)$$

where $\psi(t) = t - \tau$ for all $t \in \mathbb{R}$ with $\tau > 0$. Since T is a multivalued F-contraction, we have, for a fixed $x_0 \in X$: $x_0 \perp y$ for all $y \in X \setminus \{x_0\}$. Then $(X, \perp_g, d)$ is a generalized orthogonal complete metric space. On the other hand, it is easy to see that T is a multivalued $\perp_{\psi F}$-contraction. Furthermore, T is generalized orthogonal preserving and generalized orthogonal continuous. Therefore, T satisfies all conditions of Theorem 2. □

Corollary. *[10, Theorem 4.3] Let T be a self-mapping on a generalized orthogonal complete metric space $(X, \perp_g, d)$ such that*

i) T is an $\perp_{\psi F}$-contraction;

ii) T is a generalized $\perp_g$-preserving;

iii) T is a generalized $\perp_g$-continuous.

Then T has a fixed point.

Corollary. *[3, Theorem 3.10] Let $(X, \perp, d)$ be an O-complete orthogonal metric space. Let $T\colon X \to X$ be a self-mapping, such that:*

i) T is an $\perp_F$-contraction, that is, T is an F-contraction for all $x, y \in X$ such that $x \perp y$.

ii) T is $\perp$-preserving;

iii) T is $\perp$-continuous.

Then T has a fixed point, moreover, T is a Picard operator.

Proof. We take $\psi(t) = t - \tau$ for all $t \in \mathbb{R}$. On the other hand, since every generalized orthogonal space is an orthogonal space, we get, from Theorem 2, the desired result. □

Example 5. Let $X = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$ be the usual metric on X. Define a binary relation on X by

$$x \perp_g y \Longleftrightarrow x < y, xy \in \{x, y\}.$$

Therefore, $(X, \perp_g, d)$ is a generalized orthogonal complete metric space, and 1 is an orthogonal element. Consider the multivalued mapping $T : X \to K(X)$ defined by

$$T(x) = \begin{cases} \{1\}, & \text{if } x = 1, 2; \\ \{2, \ldots, x-1\}, & \text{if } x \geqslant 3. \end{cases}$$

Hence, T is a multivalued generalized orthogonal preserving. Now, let $F \in \mathcal{F}$ be defined by $F(t) = t + \ln t$ for all $t > 0$, and $\psi \in \Psi$ be defined by $\psi(x) = x - 1$ for all $x \in \mathbb{R}$.

Let $x, y \in X$, such that $x \perp_g y$ and $H(Tx, Ty) > 0$. We obtain $x = 1$ and $y \geqslant 3$; then

$$\frac{H(T1, Ty)}{M(1, y)} e^{H(T1,Ty)-M(1,y)} \leqslant \frac{y-2}{y-1} e^{y-2-y+1} \leqslant e^{-1}.$$

This means that T is a generalized multivalued $\perp_{\psi F}$-contraction; then all assumptions of Theorem 2 are satisfied and 1 is a fixed point.

Now, since

$$\frac{H(T4, TS_2)}{M(4, 2)} e^{H(T4,TS_2)-M(4,2)} \geqslant \frac{3}{2} > e^{-1},$$

T is not a generalized multivalued $\perp_F$-contraction.

4. Application. In this section, we give a typical application of our results to integral inclusions. Inspired by [7], [9], [10], we study the existence of a solution for a Volterra-type integral inclusion. For this purpose, let $X = \mathcal{C}([1, \theta], [1, \infty))$ be the space of all continuous functions from $I = [1, \theta]$ into $[1, \infty)$ with $\theta > 1$. Let us consider the Volterra-type inclusion

$$x(t) \in f(t) + \int_1^t K(t, s, x(s))\, ds, \ \ t \in I, \tag{17}$$

where $K \colon I \times I \times \mathbb{R}^+ \to \mathcal{P}_{cv}(\mathbb{R}^+)$ and $\mathcal{P}_{cv}(\mathbb{R}^+)$ denotes the class of nonempty compact and convex subsets of $\mathbb{R}^+$. For each $x \in X$, the multi-valued mapping $K_x(t,s) := K(t,s,x(s))$, $(t,s) \in [1,\theta]^2$ is lower semicontinuous and $f \in X$ with $f \geqslant 2$.

We can define a multivalued operator T from X into $\mathcal{P}(X)$ by

$$Tx(t) = \left\{ v \in X : v(t) \in f(t) + \int_1^t K(t,s,x(s))\,ds,\ t \in I \right\}, \tag{18}$$

for all $x \in X$.

Let $x \in X$; by Michael's selection Theorem [5], there exits a continuous operator $k_x \colon I \times I \to \mathbb{R}^+$, such that $k_x(t,s) \in K_x(t,s)$ for any $t,s \in [1,\theta]$, which implies that $f(t) + \int_1^t k_x(t,s)ds \in Tx(t)$; then $T(x) \neq \emptyset$. On the other hand, it is obvious to see that Tx is a closed set.

Now, suppose that for any $x,y \in \mathcal{C}(I)$ with $\sqrt{x(s)}y(s) > y(s)$ and for any $s \in I$ we have:

$$H(K(s,t,x(s)),K(s,t,y(s))) \leqslant e^{\frac{2}{\sqrt{\alpha(s)}}}|x(s) - y(s)|, \tag{19}$$

where α is a positive function from $\mathcal{C}(I)$ and

$$|x(s) - y(s)| \leqslant Ce^{A(s)} \leqslant \alpha(s)e^{A(s)} \tag{20}$$

for all $s \in I$, where C is a positive constant and $A(s) := \int_1^s \alpha(w)dw$.

Under the above assumptions, we have the following theorem:

Theorem 4. *Suppose that the assumption above are satisfied; then the integral inclusion* (17) *has a unique positive solution.*

Proof. Define a generalized orthogonal relation $\perp_g$ on X as follows:

$$x \perp_g y \Longleftrightarrow \sqrt{x(s)}y(s) > y(s) \quad \text{for all } s \in I. \tag{21}$$

By (21), it is clear to see that $\perp_g$ is a generalized orthogonal relation on X and $x_0 = 1$ is a generalized orthogonal element.

We provide X with the metric $d \colon X \times X \to [0,\infty)$ defined by

$$d(x,y) = \sup_{t \in I} e^{-A(t)}|x(t) - y(t)|$$

for all $x,y \in X$ (it is known that such a norm is equivalent to the standard supremum norm). Therefore, $(X,\perp_g,d)$ is a generalized orthogonal complete metric space, and, hence, condition (iv) of Theorem 2 is satisfied.

Condition (ii): T is a multivalued generalized $\perp_g$ preserving.

Let $x, y \in X$, such that $x \perp_g y$, $H(Tx, Ty) > 0$ and $t \in I$; then, for all $a \in Tx$ and $b \in Ty$, there exist $k_x \in K_x$ and $k_y \in K_y$ with

$$a(t) := f(t) + \int_1^t k_{x(s)}(t,s)\,ds = f(t) + \int_1^t k(t,s,x(s))ds \geqslant 2,$$

$$b(t) := f(t) + \int_1^t k_{y(s)}(t,s)\,ds = f(t) + \int_1^t k(t,s,y(s))\,ds \geqslant 2,$$

and, hence, $\sqrt{a(t)}b(t) > b(t)$. Then

$$Tx \perp_g Ty.$$

Condition (iii): T is a multivalued generalized $\perp_g$ continuous.

It is clear to see from the fact $Tx(t) := f(t) + \int_1^t K(s,t,x(s))\,ds$ that T is a multivalued generalized orthogonal continuous mapping.

It is obvious to see that (17) has a positive solution if only if T has a fixed point, and, hence, it remains to prove:

Condition (i): T is a multivalued $\perp_{\psi F}$ contraction.

For this, take $F(t) = -\dfrac{1}{\sqrt{t}}$ for all $t > 0$ and $\psi(z) = -e^{-z}$ for all $z \in \mathbb{R}$. It is easy to show that $F \in \mathcal{F}$ and $\psi \in \Psi$. Now, we show (5) of Definition 14. Indeed, we have $|\psi(t)| = e^{-t} \geqslant 0$ and $|\psi^2(t)| = e^{e^{-t}} \geqslant 1$. Suppose by induction that $|\psi^k(t)| \geqslant k-1$, for all $k \in \mathbb{N}$, $t \in \mathbb{R}$. Hence,

$$|\psi^{k+1}(t)| = e^{-\psi^k(t)} \geqslant e^{k-1} \geqslant k.$$

Then

$$\sum_{k \geqslant 1} |\psi^k(t)|^{-1/\lambda} \leqslant \sum_{k \geqslant 1} k^{-1/\lambda} < \infty, \text{ for all } \lambda \in (0,1).$$

Also, let $x, y \in X$ with $x \perp_g y$. Suppose that $H(Tx, Ty) > 0$; it follows from (20) that for any $s \in I$

$$|x(s) - y(s)| \leqslant Ce^{A(s)} \leqslant \alpha(s)e^{A(s)},$$

and, hence,

$$d(x,y) = \sup_{s \in I} e^{-A(s)}|x(s) - y(s)| \leqslant \alpha(s). \tag{22}$$

As $t \mapsto e^{\frac{2}{\sqrt{t}}}$ is a decreasing function on I, we obtain the form (22):

$$e^{\frac{2}{\sqrt{\alpha(s)}}} \leqslant e^{\frac{2}{\sqrt{d(x,y)}}}. \tag{23}$$

Now, let $u \in Tx$, so there exists $k_x(t,s) \in K_x(t,s)$ for $t, s \in [1,\theta]$ with $u(t) = f(t) + \int_1^t k_x(t,s)\,ds$. On the other hand, condition (19) implies that there exists $v(t,s) \in K_y(t,s)$, such that

$$|k_x(t,s) - v(t,s)| \leqslant e^{\frac{2}{\sqrt{\alpha(s)}}}|x(s) - y(s)|,$$

for all $t, s \in [1,\theta]$.

We define a multivalued operator S by

$$S(t,s) = K_y(t,s) \cap \{w \in \mathbb{R} : |k_x(t,s) - w| \leqslant e^{\frac{2}{\sqrt{\alpha(s)}}}|x(s) - y(s)|\},$$

for all $t, s \in [1,\theta]$.

On the other hand, S is lower semicontinuous; it follows that there exists a continuous mapping $k_y \colon [1,\theta]^2 \to [1,\infty)$, such that $k_y(t,s) \in S(t,s)$, for all $t, s \in [1,\theta]$ (see [7], [9]). Then we have

$$z(t) = f(t) + \int_1^t k_y(t,s)\,ds \in f(t) + \int_1^t K(t,s,y(s))\,ds, \;\; t \in [1,\theta]$$

and for all $t \in [1,\theta]$ we obtain

$$|u(t) - z(t)| = \left| \int_1^t k_x(t,s)\,ds - \int_1^t k_y(t,s)\,ds \right| \leqslant$$

$$\leqslant \int_0^t |k_x(t,s) - k_y(t,s)|\,ds \leqslant \int_1^t e^{\frac{2}{\sqrt{\alpha(s)}}}|x(s) - y(s)|\,ds, \tag{24}$$

From (22), (23) and (24), we get:

$$|u(t) - z(t)| \leqslant e^{\frac{2}{\sqrt{d(x,y)}}} \int_1^t |x(s) - y(s)|\,ds \leqslant$$

$$\leqslant e^{\frac{2}{\sqrt{d(x,y)}}} \int_1^t |x(s)-y(s)| e^{-A(s)} e^{A(s)}\, ds \leqslant$$

$$\leqslant d(x,y) e^{\frac{2}{\sqrt{d(x,y)}}} \int_1^t \frac{1}{\alpha(s)} \alpha(s) e^{A(s)} ds \leqslant e^{\frac{2}{\sqrt{d(x,y)}}} \left[e^{A(t)} - 1\right].$$

Then

$$e^{-A(t)} |u(t) - z(t)| \leqslant e^{\frac{2}{\sqrt{d(x,y)}}}.$$

On the other hand, by interchanging the roles of x, y, and using Lemma 2, we obtain

$$H(Tx, Ty) \leqslant e^{\frac{2}{\sqrt{d(x,y)}}}.$$

Thus,

$$\sqrt{H(Tx,Ty)} \leqslant e^{\frac{1}{\sqrt{d(x,y)}}}.$$

Therefore,

$$e^{\frac{-1}{\sqrt{d(x,y)}}} \leqslant \frac{1}{\sqrt{H(Tx,Ty)}}.$$

Hence,

$$-\frac{1}{\sqrt{H(Tx,Ty)}} \leqslant -e^{\frac{-1}{\sqrt{d(x,y)}}}.$$

At the end, we have

$$F\big(H(Tx,Ty)\big) \leqslant \psi\big[F\big(d(x,y)\big)\big].$$

Theorem 2 implies that the integral inclusion (17) has a positive solution. □

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