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REGULAR GROWTH OF DIRICHLET SERIES OF THE CLASS $D(\Phi)$ ON CURVES OF BOUNDED K-SLOPE

Abstract. We study the asymptotic behavior of the sum of entire Dirichlet series with positive exponents on curves of a bounded slope going in a certain way to infinity. For entire transcendental functions of finite order, Polia showed that if the density of the sequence of exponents is equal to zero, then for any curve going to infinity there is an unbounded sequence of points on which the logarithm of the modulus of the sum of the series is equivalent to the logarithm of the maximum of the modulus. Later, these results were completely transferred by I. D. Latypov to entire Dirichlet series of finite order and finite lower order by Ritt. Further generalization was obtained in the works of N. N. Yusupova–Aitkuzhina to the more general dual classes of Dirichlet series defined by the convex majorant. In this paper, we obtain necessary and sufficient conditions for the exponents under which the logarithm of the modulus of the sum of any Dirichlet series from one such class on a curve of bounded slope is equivalent to the logarithm of the maximum term on an asymptotic set whose upper density is not less than a positive number depending only on the curve.

Key words: *Dirichlet series, maximal term, the curve of a bounded slope, asymptotic set*

2020 Mathematical Subject Classification: *30D10*

1. Introduction. The problem investigated here goes back to the well-known Polya problem [12].

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{p_n} \tag{1}$$

be an entire transcendental function, $P = \{p_n\}$ be a sequence of natural numbers having a density Δ, i.e.,

$$\lim_{n\to\infty} \frac{n}{p_n} = \Delta(P) := \Delta$$

exists.
Polya [12] showed that if $\Delta = 0$, then in each angle $\{z\colon |\arg(z-\alpha)| \leqslant \delta\}$, $\delta > 0$, the function f has the same order as in the whole plane. The corresponding result for Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad 0 < \lambda_n \uparrow \infty, \tag{2}$$

absolutely converging in the whole plane, is proved in [13]: if for a sequence $\Lambda = \{\lambda_n\}$ the conditions $\Delta = 0$ and $\lambda_{n+1} - \lambda_n \geqslant h > 0$, $n \geqslant 1$, are satisfied, then R-order of the function F on the positive ray $\mathbb{R}_+ = [0, \infty)$ is equal to R-order ρ_R of the function F in the whole plane. A more general result is proved in [2], where, in particular, it is shown that if $\Delta = 0$ and the condensation index

$$\delta = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \ln \left|\frac{1}{Q'(\lambda_n)}\right|, \quad Q(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right),$$

of the sequence Λ is equal to zero, then $\rho_R = \rho_\gamma$, where

$$\rho_\gamma = \overline{\lim_{s\in\gamma,\, s\to\infty}} \frac{\ln\ln|F(s)|}{\sigma}, \quad \sigma = \operatorname{Re} s,$$

is the Ritt order on the curve γ going to infinity, so that if $s \in \gamma$ and $s \to \infty$, then $\operatorname{Re} s \to +\infty$.

The most general, but somewhat different result is established in the article [7]. In order to formulate it, we will introduce the appropriate notation and definitions.

Let $\Gamma = \{\gamma\}$ be a family of all curves going to infinity so that if $s \in \gamma$ and $s \to \infty$, then $\operatorname{Re} s \to +\infty$.

We denote by $D(\Lambda)$ the class of entire functions F representable as Dirichlet series (2) in the whole plane, and by $D(\Lambda, R)$ a subclass of $D(\Lambda)$ consisting of functions F having finite order $\rho_R(F)$ by Ritt:

$$\rho_R(F) = \overline{\lim_{\sigma\to+\infty}} \frac{\ln\ln M_F(\sigma)}{\sigma}, \quad M_F(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)|.$$

For $F \in D(\Lambda)$, $\gamma \in \Gamma$, we assume

$$d(F;\gamma) \stackrel{def}{=} \varlimsup_{s\in\gamma,\, s\to\infty} \frac{\ln|F(s)|}{\ln M_F(\operatorname{Re}s)}, \quad d(F) = \inf_{\gamma\in\Gamma} d(F;\gamma).$$

Denote by L the class of all continuous and unboundedly increasing positive functions on $[0,\infty)$.

The sequence $\{b_n\}$ ($b_n \neq 0$ for $n \geqslant N$) is called $\overline{W}$-normal (more precisely, $W(\ln)$-normal) if there is a function $\theta \in L$, such that [7]

$$\lim_{x\to\infty} \frac{1}{\ln x} \int\limits_1^x \frac{\theta(t)}{t^2} dt = 0, \qquad -\ln|b_n| \leqslant \theta(\lambda_n), \quad n \geqslant N.$$

The Weierstrass product

$$Q(z) = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{\lambda_n^2}\Big), \quad 0 < \lambda_n \uparrow \infty$$

is known to be an entire function of the exponential type if and only if the sequence Λ has a finite upper density

$$\overline{\Delta} := \overline{\Delta}(\Lambda) = \varlimsup_{n\to\infty} \frac{n}{\lambda_n} < \infty.$$

In [7] the following result is proved:

Theorem 1. *Let the sequence Λ have finite upper density. Assume that the sequence $\{Q'(\lambda_n)\}$ is $\overline{W}$–normal. In order for the equality $d(F) = 1$ to be valid for every function $F \in D(\Lambda, R)$, the condition*

$$\varliminf_{x\to\infty} \frac{1}{\ln x} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0 \tag{3}$$

is necessary and sufficient.

Let the entire function f of finite order have the form (1). If the sequence P has density $\Delta = 0$, then $d(f) = 1$ ($d(f)$ is an analog of the value $d(F)$, which is determined by all possible curves arbitrarily going to infinity). This fact was first established by Polia in [12]. Note that the equality $d(f) = 1$ follows from the more general Theorem 1. Indeed, since $\Delta = 0$, then, obviously,

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{p_n \leqslant x} \frac{1}{p_n} = 0.$$

Since $\Delta = 0$, and $p_n \in \mathbb{N}$, then, as it is known, the condensation index (see, for example, [11])

$$\delta = \lim_{n\to\infty} \frac{1}{p_n} \ln \left| \frac{1}{Q'(p_n)} \right| = 0.$$

This means that there exists a function $\theta \in L$, $\theta(x) = o(x)$ for $x \to \infty$, such that

$$-\ln |Q'(p_n)| \leqslant \theta(p_n), \quad n \geqslant 1.$$

So, the sequence $\{Q'(p_n)\}$ is $\overline{W}$-normal ($W(\ln)$-normal).

Finally, if f is an entire function of finite order, then, assuming $z = e^s$, we notice that

$$F(s) = f(e^s) = \sum_{n=1}^{\infty} a_n e^{p_n s}$$

is an entire function of finite R-order. Therefore, $d(f) = d(F)$, and everything follows from Theorem 1.

However, from the fact that $d(F) = 1$, generally speaking, it does not follow that the equality $\rho_R(F) = \rho_\gamma$ is fulfilled for the Ritt orders of the function F in the whole plane and on the curve $\gamma \in \Gamma$. It turns out that if in Theorem 1 the condition (3) is replaced by a stronger requirement

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0,$$

then $\rho_R(F) = \rho_\gamma$ for any function $F \in D(\Lambda, R)$ (see [8]).

As in [8], a more general situation is considered here; namely, the class of Dirichlet series (2) defined by some convex growth majorant is studied. For curves $\gamma \in \Gamma$ having a bounded slope, a stronger asymptotic estimate than the equality $d(F) = 1$ obtained in [8] for functions from the same class, is proved.

By definition, the curve $\gamma \in \Gamma$, given by the equation $y = g(x)$, $x \in \mathbb{R}_+ = [0, +\infty)$, has a bounded slope if

$$\sup_{\substack{x_1, x_2 \in \mathbb{R} \\ x_1 \neq x_2}} \left| \frac{g(x_2) - g(x_1)}{x_2 - x_1} \right| = K < \infty. \tag{4}$$

Condition (4) means that absolute values of tangents of all chords of the curve γ do not exceed K. In this case, γ is called a curve of the bounded

K-slope. In a number of papers, a close relationship was found between the regularity of the growth of the sum of the Dirichlet series (2) on $\gamma \in \Gamma$ and the incompleteness of the exponent system $\left\{e^{\lambda_n z}\right\}$ on arcs $\gamma' \subset \gamma$, especially with the strong incompleteness of this exponential system in the vertical strip (see [3], [6], [9]). It should be noted that the results of [6], [9] on the incompleteness of the system $\left\{e^{\lambda_n z}\right\}$ on arcs can be applied to the study of uniqueness theorems and asymptotic properties of entire Dirichlet series (2) without any restrictions on the growth of $M_F(\sigma)$, i.e., in the most general case.

The purpose of this paper is to show that, under the same conditions on Λ as in [8], if

$$\varlimsup_{\sigma \to +\infty} \frac{\ln M_F(\sigma)}{\Phi(\sigma)} < \infty$$

(Φ is some convex function on $\mathbb{R}_+$), then for any curve $\gamma \in \Gamma$ of bounded K-slope at $s \in \gamma$, $\sigma = \operatorname{Re} s \to +\infty$ by some asymptotic set $A \subset \mathbb{R}_+$, whose upper density is $DA \geqslant 1/\sqrt{K^2+1}$, the asymptotic equality of the Polya

$$\ln |F(s)| \sim \ln M_F(\sigma), \quad s \in \gamma, \quad \sigma = \operatorname{Re} s \to +\infty,$$

is valid.

It is clear that this relationship is significantly better than the equality $d(F) = 1$.

2. Auxiliary statements. Main results. Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence having finite upper density D. Then $Q(z)$ is an entire function of exponential type of not higher than πD^*, where D^* is the averaged upper density of the sequence Λ:

$$D^* = \varlimsup_{t \to \infty} \frac{N(t)}{t}, \quad N(t) = \int_0^t \frac{n(x)}{x}\, dx, \quad n(t) = \sum_{\lambda_j \leqslant t} 1.$$

Always $D^* \leqslant D \leqslant eD^*$ (see, for example, [11], [10]).

Let L be the class of all continuous and unboundedly increasing on $\mathbb{R}_+$ positive functions, Φ be a convex function of L,

$$D_m(\Phi) = \left\{F \in D(\Lambda)\colon \ln M_F(\sigma) \leqslant \Phi(m\sigma)\right\}, \quad m \in \mathbb{N},$$

where $M_F(\sigma) = \sup\limits_{|t|<\infty} |F(\sigma + it)|$. Let

$$D(\Phi) = \bigcup_{m=1}^{\infty} D_m(\Phi).$$

Suppose that the function Φ introduced above is such that

$$\overline{\lim_{x\to\infty}} \frac{\varphi(x^2)}{\varphi(x)} < \infty, \tag{5}$$

where φ is the inverse function of Φ. For our purposes, we need the following class of monotonic functions:

$$W(\varphi) = \Big\{ w \in L\colon \sqrt{x} \leqslant w(x),\, \lim_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_1^x \frac{w(t)}{t^2} dt = 0 \Big\}.$$

Note that the restriction $\sqrt{x} \leqslant w(x)$ in this definition does not limit the generality. It is introduced for convenience only. Let $\Gamma = \{\gamma\}$ be the family of curves γ introduced above, and let, for $F \in D(\Lambda)$,

$$d(F;\gamma) \overset{def}{=} \overline{\lim_{s\in\gamma,\, s\to\infty}} \frac{\ln|F(s)|}{\ln M_F(\mathrm{Re} s)}, \quad d(F) = \inf_{\gamma\in\Gamma} d(F;\gamma). \tag{6}$$

By $\mu(\sigma)$ denote the maximal term of the series (2):

$$\mu(\sigma) = \max_{n\geqslant 1}\{|a_n|e^{\lambda_n \sigma}\}, \quad \sigma = \mathrm{Re}\, s.$$

In [8], the criterion of equality $d(F) = 1$ is proved for any function F from the class $\underline{D}(\Phi)$, where

$$\underline{D}(\Phi) = \bigcup_{m=1}^{\infty} \underline{D}_m(\Phi),$$

$$\underline{D}_m(\Phi) = \{F \in D(\Lambda)\colon \exists\{\sigma_n\}\colon 0 < \{\sigma_n\} \uparrow \infty,\ \ln M_F(\sigma_n) \leqslant \Phi(m\sigma_n)\},\ m \geqslant 1.$$

We will say that the sequence $\{Q'(\lambda_n)\}$ is $W(\varphi)$–normal if there exists $\theta \in L$, such that

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_1^x \frac{\theta(t)}{t^2} dt = 0, \qquad -\ln\big|Q'(\lambda_n)\big| \leqslant \theta(\lambda_n), \quad n \geqslant 1. \tag{7}$$

In [8] the following result is proved:

Theorem 2. *Let the sequence Λ have finite upper density. Assume that the sequence $\{Q'(\lambda_n)\}$ is $W(\varphi)$-normal.*

For the equality $d(F) = 1$ to be valid for any function $F \in \underline{D}(\Phi)$, it is necessary and sufficient for the condition

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0 \tag{8}$$

to be valid.

For functions $F \in D(\Phi)$ on condition

$$\varliminf_{x\to\infty} \frac{1}{\varphi(x)} \sum_{\lambda_n \leqslant x} \frac{1}{\lambda_n} = 0$$

the dual theorem is also true [8].

Note that in the definition of the class $D(\Phi)$ you can, for example, consider the function

$$\Phi(\sigma) = \underbrace{\exp\exp\ldots\exp}_{k}(\sigma), \quad k \geqslant 1.$$

Hence, from Theorem 2 the corresponding result from [7] follows, proved for the case of $k = 1$.

Let us formulate the main result. To do this, we introduce another class of functions.

Let Φ be the function defined above, φ be the inverse function of Φ,

$$\underline{W}(\varphi) = \left\{ w \in L\colon\ \sqrt{x} \leqslant w(x),\ \lim_{x\to\infty} \frac{w(x)}{x\varphi(x)} = 0,\ \varliminf_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w(t)}{t^2} dt = 0 \right\}.$$

The following result is valid:

Theorem 3. *Let the upper density of the sequence Λ be finite, and the sequence $\{Q'(\lambda_n)\}$ be $W(\varphi)$-normal. If condition (8) holds, then for any function $F \in D(\Phi)$ and for any curve $\gamma \in \Gamma$ of a bounded K-slope the asymptotic equality*

$$\ln|F(s)| = (1+o(1)) \ln M_F(\sigma), \quad s \in \gamma, \quad \operatorname{Re} s = \sigma \in A, \quad \sigma \to +\infty, \tag{9}$$

is valid. Here $A \subset \mathbb{R}_+$ is such that

$$DA = \varlimsup_{\sigma\to+\infty} \frac{\operatorname{mes}(A \cap [0,\sigma])}{\sigma} \geqslant \frac{1}{\sqrt{1+K^2}}.$$

Here are the lemmas used to prove Theorem 3.

Lemma 1. *Let $\Phi \in L$, and for the function φ, the inverse of Φ, condition (5) be satisfied. Let, further, $u(\sigma)$ be non-decreasing, positive, and continuous on $[0, \infty)$, at that $\lim_{\sigma\to\infty} u(\sigma) = \infty$, and for some $m \in \mathbb{N}$ the estimate* [1]

$$u(\sigma) \leqslant \ln \Phi(m\sigma)$$

is valid.

Suppose that the function w belongs to the class $\underline{W}(\varphi)$, and for the sequence $\{x_n\}$, $0 < x_n \uparrow \infty$,

$$\lim_{n\to\infty} \frac{1}{x_n} \int_1^{x_n} \frac{w(t)}{t^2} dt = 0.$$

If $v = v(\sigma)$ is the solution of the equation

$$w(v) = e^{u(\sigma)},$$

and the numbers τ_n are the roots of the equation $v(\tau) = x_n$, $n \geqslant 1$, then for $\sigma \to \infty$ outside of some set $E \subset [0, \infty)$,

$$\operatorname{mes}(E \cap [0, \tau_n]) = o(\varphi(v(\tau_n))),\ \tau_n \to \infty,$$

an estimate

$$u\left(\sigma + \frac{w(v(\sigma))}{v(\sigma)}\right) < u(\sigma) + o(1)$$

is valid.

Lemma 1 is proved in [1].

Lemma 2. *Let the function $g(z)$ be analytic and bounded in a circle $D(0, R) = \{z\colon |z| < R\}$, $|g(0)| \geqslant 1$. If $0 < r < 1 - N^{-1}$, $N > 1$, then there is no more than a countable set of circles*

$$V_n = \{z\colon |z - z_n| \leqslant \rho_n\}, \quad \sum_n \rho_n \leqslant Rr^N(1 - r), \tag{10}$$

[1] In [1], lemma 1 was proved when the estimate $u(\sigma) \leqslant C\Phi(\sigma)$ is valid, although only the inequality $u(\sigma) \leqslant \Phi(m\sigma)$ is used in its proof. Therefore, the formulation of Lemma 1 is given here in a more general form.

such that for all z from the circle $\{z\colon |z| \leqslant rR\}$, but outside of $\bigcup\limits_n V_n$, the estimate

$$\ln|g(z)| \geqslant \frac{R-|z|}{R+|z|}\ln|g(0)| - 5NL, \tag{11}$$

is valid, where

$$L = \frac{1}{2\pi}\int\limits_0^{2\pi} \ln^+\left|g(Re^{i\theta})\right| d\theta - \ln|g(0)|.$$

Lemma 2 is proved in [4].

3. Proof of Theorem 3.

Sequence $\{Q'(\lambda_n)\}$ is $W(\varphi)$–normal, and $\Lambda = \{\lambda_n\}$ has finite upper density. Therefore,

$$\varlimsup_{x\to\infty}\frac{N(x)}{x} < \infty, \quad -\ln|Q'(\lambda_n)| \leqslant \theta(\lambda_n), \, n \geqslant 1, \, \theta \in W(\varphi).$$

Since (see [8])

$$\sup_{x>0}\left|\sum_{\lambda_n\leqslant x}\frac{1}{\lambda_n} - \int\limits_0^x \frac{N(t)}{t^2}\right| = a < \infty,$$

then, taking into account (7), (8) from here we get

$$\lim_{x\to\infty}\frac{1}{\varphi(x)}\int\limits_0^x \frac{N(t)}{t^2}dt = 0.$$

Let us assume that $w(t) = \max(\sqrt{t}, N(et) + \theta(t))$, where θ is a function from condition (7). It is clear that $w \in \underline{W}(\varphi)$. Then, obviously, there is a function $w^* \in \underline{W}(\varphi)$, such that $w^*(x) = \beta(x)w(x), \quad \beta \in L$.

Let $v = v(\sigma)$ be the solution of the equation

$$w^*(v) = 3\ln\mu(\sigma). \tag{12}$$

Let us assume that

$$h = \frac{w(v(\sigma))}{v(\sigma)}, \quad h^{(1)} = \frac{w_1(v)}{v}, \quad h^* = \frac{w^*(v(\sigma))}{v(\sigma)},$$

where $w_1(v) = \sqrt{\beta(x)}w(x)$. Let

$$R_v = \sum_{\lambda_j > v} |a_j|\, e^{\lambda_j \sigma}, \quad v = v(\sigma).$$

Since the sequence Λ has finite upper density, then $C = \sum\limits_{n=1}^{\infty} \lambda_n^{-2} < \infty$. Therefore, the estimate (see, for example, [3])

$$R_v \leqslant C\mu(\sigma + h^*) \exp\left[-(1 + o(1))w^*(v)\right] \tag{13}$$

is correct.

Consider the function $u(\sigma) = \ln 3 + \ln\ln \mu(\sigma)$. Since $\mu(\sigma) \leqslant M_F(\sigma)$ and $F \in D(\Phi)$, then there is $m \geqslant 1$, such that

$$u(\sigma) \leqslant \ln \Phi(m\sigma).$$

Therefore, taking into account (12), we have:

$$\ln w^*(v(\sigma)) = u(\sigma) \leqslant \ln \Phi(m\sigma), \;\; m \geqslant 1.$$

Also,

$$\frac{1}{\sigma} \leqslant \frac{m}{\varphi(w^*(v(\sigma)))}, \quad m \geqslant 1. \tag{14}$$

Taking into account condition (5) and the fact that $\sqrt{x} \leqslant w^*(x)$, we have:

$$\varphi(x) \leqslant C_1 \varphi(w^*(x)), \quad x \geqslant x_0, \quad 0 < C_1 < \infty. \tag{15}$$

As a result, from (14) and (15) we get the estimates

$$\frac{1}{\sigma} \leqslant \frac{C_2}{\varphi(v(\sigma))}, \quad 0 < C_2 < \infty. \tag{16}$$

Further, since $w^* \in \underline{W}(\varphi)$, then

$$\lim_{x \to \infty} \frac{w^*(x)}{x\varphi(x)} = 0, \tag{17}$$

and for some sequence $\{x_n\}$, $0 < x_n \uparrow \infty$:

$$\lim_{n \to \infty} \frac{1}{\varphi(x_n)} \int_1^{x_n} \frac{w^*(t)}{t^2}\, dt = 0. \tag{18}$$

Applying Lemma 1 for the functions u and w^* and considering (16), as well as the definition of the numbers τ_j, $j \geqslant 1$, for $\sigma \to \infty$, outside of some set $E_1 \subset [0, \infty)$:

$$\operatorname{mes}(E_1 \cap [0, \tau_j]) \leqslant o(\varphi(v(\tau_j))) = o(\tau_j),\ \tau_j \to \infty, \tag{19}$$

we get

$$\mu(\sigma + 3h^*(\sigma)) = \mu(\sigma)^{1+o(1)}. \tag{20}$$

Hence, from (13), (20) we learn that when $\sigma \to \infty$,

$$R_v \leqslant C\mu(\sigma)^{1+o(1)} \exp\left[-w^*(v)(1 + o(1))\right] = \mu(\sigma)^{-2(1+o(1))} \tag{21}$$

outside of E_1. It follows that $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$ for $\sigma \geqslant \sigma_1$, $\sigma \notin E_1$, where $\lambda_{\nu(\sigma)}$ is the central exponent ($\nu(\sigma)$ is the central index) of the series (2).

In the same way as (21), it is shown that when $\sigma \to \infty$, then outside of the same set E_1 (see [6]):

$$\sum_{\lambda_j > v(\sigma)} |a_j| e^{\lambda_j(\sigma + h^{(1)})} \leqslant \mu^{-2(1+o(1))}(\sigma). \tag{22}$$

The Borel-Nevanlinna ratio (20) allows us to do this, since $h^{(1)}(\sigma) = $ $= o(h^*(\sigma))$ for $\sigma \to \infty$ (properties (17), (18) are necessary when proving Lemma 1).

Let

$$F_a(s) = \sum_{\lambda_n \leqslant a} a_n e^{\lambda_n s}, \qquad s = \sigma + it.$$

Then, for $\lambda_n \leqslant a$, we have (see [11]):

$$a_n = e^{-\alpha\lambda_n} \frac{1}{2\pi i} \int\limits_C \varphi_n(t) F_a(t + \alpha) dt, \tag{23}$$

where α is an arbitrary parameter,

$$\varphi_n(t) = \frac{1}{Q_a'(\lambda_n)} \int\limits_0^\infty \frac{Q_a(\lambda)}{\lambda - \lambda_n} e^{-\lambda t} d\lambda, \qquad Q_a(\lambda) = \prod_{\lambda_n \leqslant a} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right), \tag{24}$$

and C is any closed contour, covering $\overline{D}$: the conjugate diagram of $Q_a(\lambda)$. But $Q_a(\lambda)$ is the polynomial, therefore, $\overline{D} = \{0\}$.

We assume $a = v(\sigma)$, $\alpha = \sigma + it$, where t is such that $\alpha \in \gamma$. As for C, lets us take the contour $\{t\colon |t| = h^{(1)}\}$, where $h^{(1)} = h^{(1)}(\sigma) = \frac{h^*(\sigma)}{\sqrt{\beta(v(\sigma))}}$. Further, by assumption,

$$-\ln|Q'(\lambda_n)| \leqslant \theta(\lambda_n) \leqslant w(\lambda_n), \quad n \geqslant 1.$$

Therefore, taking into account equality (12), we obtain that for all $\lambda_n \leqslant v(\sigma)$ for $\sigma \to \infty$ we have

$$\frac{1}{|Q_v'(\lambda_n)|} \leqslant \frac{1}{|Q'(\lambda_n)|} \leqslant e^{\theta(\lambda_n)} \leqslant e^{w(v)} = e^{o(w^*(v))} = \mu(\sigma)^{o(1)}.$$

Then, from (23), (24) we see that for all $\lambda_n \leqslant v(\sigma)$ for $\sigma \to \infty$ outside of E_1

$$|a_n|e^{\lambda_n\sigma} \leqslant$$
$$\leqslant \mu(\sigma)^{o(1)}h^{(1)}\Big[\max_{|\xi-\alpha|\leqslant h^{(1)}}|F(\xi)| + \sum_{\lambda_j > v}|a_j|\, e^{\lambda_j(\sigma+h^{(1)})}\Big]\int_0^\infty\Big|\frac{Q_v(\lambda)}{\lambda-\lambda_n}\Big||e^{-\lambda t}||d\lambda|, \tag{25}$$

where $\alpha = \sigma + it \in \gamma$.

It is not difficult to show that (see [5])

$$\max_{|\lambda|=r}\Big|\frac{Q_v(\lambda)}{\lambda-\lambda_n}\Big| \leqslant M(1)M_v(r), \tag{26}$$

where $M(1) = \max\limits_{|z|=1}|Q(z)|$, $M_v(r) = \max\limits_{|z|=r}|Q_v(z)|$.

Since $\lambda_\nu(\sigma) \leqslant v(\sigma)$ outside of E_1 for $\sigma \geqslant \sigma'$, taking into account (22), (26), from (25) for $\sigma \to \infty$ outside of E_1 we get:

$$\mu(\sigma)^{1+o(1)} \leqslant h^{(1)}[\max_{|\xi-\alpha|\leqslant h^{(1)}}|F(\xi)| + \mu(\sigma)^{-2(1+o(1))}]\int_0^\infty M_v(r)e^{-rh^{(1)}}dr. \tag{27}$$

Further, taking into account the definitions of the quantities $v = v(\sigma)$, $h^{(1)} = h^{(1)}(\sigma)$, and also inequalities $n(x) \leqslant N(ex)$, $\ln(1+x^2) < x$, $x > 0$, we have:

$$\ln M(r) = n(v)\ln\left(1+\frac{r^2}{v^2}\right) + 2r^2\int\limits_0^v \frac{n(t)}{t(t^2+r^2)}dt \leqslant$$
$$\leqslant \frac{n(v)}{v}r + 2N(v) = o(1)h^{(1)}r + o(1)\ln\mu(\sigma).$$

Hence, from (27) we obtain that for $\sigma\to\infty$ outside E_1

$$\mu(\sigma)^{1+o(1)} \leqslant \max_{|\xi-\alpha|\leqslant h^{(1)}} |F(\xi)| = |F(\xi^*)|, \tag{28}$$

where $|\xi^*-\alpha| = h^{(1)}$, $\alpha=\sigma+it\in\gamma$. Taking into account the estimate (26), for $\sigma\to\infty$ outside of E_1 we also have

$$\mu(\sigma)\leqslant M_F(\sigma)\leqslant M_F(\sigma+2h^*)\leqslant \sum_{n=1}^{\infty}|a_n|e^{\lambda_n(\sigma+2h^*)}\leqslant$$
$$\leqslant \mu(\sigma+3h^*)\Big[n(v)+\sum_{\lambda_j>v(\sigma)} e^{-h^*\lambda_j}\Big] < \mu(\sigma)^{1+o(1)}. \tag{29}$$

Let $B=\mathbb{R}_+\backslash E_1$, $h=w(v(\sigma))/v(\sigma)$. Then there is a sequence $\{\sigma_j\}$, $\sigma_j\in B$, $\sigma_j\uparrow 0$, $\sigma_j+h_j\leqslant\sigma_{j+1}$, $j\geqslant 1$, such that (see [4])

$$B\subset\bigcup_{j=1}^{\infty}\left[\sigma_j-h_j,\sigma_j+h_j\right],$$

where $h_j=w(v_j)/v_j$, $v_j=v(\sigma_j)$, $j\geqslant 1$.

Let us assume that $g(z) = F(z+\xi^*)$. From (28) it can be seen that $|g(0)|\geqslant 1$ when $\sigma\geqslant\sigma''>\sigma'$ outside of E_1. We apply Lemma 1 to the function $g(z)$, assuming in (28) that $h^{(1)}=h_j^{(1)}=\frac{w(v_j)}{v_j}\sqrt{\beta(v_j)}$, $\alpha_j=\sigma_j+it_j$, and assuming in the estimates (10), (11) that $N=4$, $r=1/\sqrt{\beta(v_j)}$, $R=h_j^*$, where $h_j^*=w^*(v_j)/v_j$, $j\geqslant j_1$. Then in the circle $\{z\colon |z|\leqslant h_j^{(1)}\}$, but outside of the exceptional circles V_{nj} with the total sum of radii

$$\sum_n \rho_n\leqslant\frac{h_j}{\beta_j},\quad \beta_j=\beta(v(\sigma_j)),\ j\geqslant j_1, \tag{30}$$

evaluation (11) is valid. Since the circle $K_j=\{z\colon |z|\leqslant h_j\}$ is contained in the circle $\{z\colon |z|\leqslant h_j^{(1)}\}$, then for all $z\in K_j$, but outside the circles

V_{nj} with the total sum of radii satisfying the estimate (30), for $j \to \infty$ we get

$$\ln|g(z)| \geqslant \left[1 + o(1) - \frac{20L}{\ln|g(0)|}\right] \ln|g(0)|. \tag{31}$$

Taking into account (28), (29), as well as the fact that $|g(0)| \geqslant 1$, we are convinced that for $j \to \infty$, an asymptotic equality

$$\frac{L}{\ln|g(0)|} = o(1),$$

holds, where

$$L = \frac{1}{2\pi}\int_0^{2\pi} \ln^+|g(Re^{i\theta})|d\theta - \ln|g(0)|,$$

$$g(0) = F(\xi^*), \quad |\operatorname{Re}\xi^* - \sigma_j| \leqslant h^{(1)}, \quad \alpha_j = \sigma_j + it_j \in \gamma.$$

Hence, from (31) for all z from the circle $\{z\colon |z| \leqslant h_j\}$, but outside the circles V_{nj} for $j \to \infty$ we have:

$$\ln|g(z)| \geqslant (1 + \mathrm{o}(1)) \ln|g(0)|. \tag{32}$$

But then, taking into account that $g(z) = F(z + \xi^*)$, and using estimates (28)–(32), we get that for all z from the circle $\{z\colon |z - \alpha_j| \leqslant h_j\}$, $\alpha_j = \sigma_j + it_j$, but outside of exceptional circles V_{nj} with the total sum of radii no more than $\frac{h_j}{\beta_j}$,

$$\ln|F(z)| > (1 + \mathrm{o}(1)) \ln\mu(\sigma_j), \ \ j \to \infty. \tag{33}$$

Let E_2 be the projection of all exceptional circles of the set $\bigcup\limits_j\{z\colon |z - \alpha_j| \leqslant h_j\}$ on B, where $\alpha_j = \sigma_j + it_j$, $B \subset \bigcup\limits_{j=1}^{\infty}[\sigma_j - h_j, \sigma_j + h_j]$, $\sigma_j \in B$, $\sigma_j + h_j \leqslant \sigma_{j+1}$, $j \geqslant 1$. Let us show that $DE_2 = 0$. Indeed, let $\sigma_j \leqslant \sigma < \sigma_{j+1}$. According to (17), $h_j \leqslant h_j^{(1)} < h_j^* = o(\sigma_j)$, $j \to \infty$. And since $\beta_j \uparrow \infty$ for $j \to \infty$, then, obviously,

$$\lim_{\sigma\to\infty} \frac{\operatorname{mes}(E_2 \cap [0, \sigma])}{\sigma} = 0.$$

So, $DE_2 = 0$, and, therefore, $dE = 0$, where $E = E_1 \cup E_2$.

The estimate (33) takes place in each circle $K_j = \{z\colon |z-\alpha_j| \leqslant h_j\}$, $\alpha_j = \sigma_j + it_j \in \gamma$, but outside the exceptional circles V_n, the total sum of whose radii satisfies the estimate (30).

Let us estimate the measure of the projection p_j of the arc $\gamma_j = \gamma \cup K_j$ on the segment $[\sigma_j - h_j, \sigma_j + h_j]$. Denoting the right-hand end of the arc γ_j by $\eta + i\mu$ (it lies on the circle ∂K_j), we have:

$$h_j^2 = (\eta - \sigma_j)^2 + [g(\eta) - g(\sigma_j)]^2 \leqslant (K^2+1)(\eta-\sigma_j)^2$$

(we assume that the arc γ of a bounded K-slope is given by the equation $y = g(x)$, $x \in \mathbb{R}_+$). As you can see, the projection length of γ_j on $[\sigma_j, \sigma_j + h_j]$ is not less than $h_j/\sqrt{K^2+1}$. The same is true for the projection of γ_j on $[\sigma_j - h_j, \sigma_j]$. So,

$$\operatorname{mes} p_j \geqslant \frac{2}{\sqrt{K^2+1}} h_j.$$

It follows that the upper density DP of the set $P = \bigcup\limits_{j=1}^{\infty} p_j$ is not less than $1/\sqrt{K^2+1}$.

Let $A = P \setminus E$. On this set, the asymptotic estimates (29), (33) are valid (A is called an asymptotic set). It follows that when $s \in \gamma$, $\operatorname{Re} s = \sigma \to \infty$ by the set A

$$\ln|F(s)| = (1+o(1)) \ln \mu(\sigma) = (1+o(1)) \ln M_F(\sigma).$$

It remains to estimate DA. We have:

$$\begin{aligned} DA = \varlimsup_{\sigma\to\infty} \frac{\operatorname{mes}(A \cap [0,\sigma])}{\sigma} \geqslant \\ \geqslant \varlimsup_{\tau_j\to\infty} \frac{\operatorname{mes}(P \cap [0,\tau_j])}{\tau_j} - \varlimsup_{\tau_j\to\infty} \frac{\operatorname{mes}(E \cap [0,\tau_j])}{\tau_j} \geqslant \frac{1}{\sqrt{K^2+1}}. \end{aligned}$$

Here $\{\tau_j\}$ is the sequence defined above.

Theorem 3 is proved.

The conditions of Theorem 3 are also necessary so that for any function $F \in D(\Phi)$ on some set $A \subset \mathbb{R}_+$ having positive upper density DA, the asymptotic equality (9) was fulfilled (see [8]).

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