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PLANAR HARMONIC MAPPINGS WITH A GIVEN JACOBIAN

Abstract. The article is devoted to the study of the Jacobians of sense-preserving harmonic mappings in the unit disk of the complex plane. The main result is a criterion for an infinitely differentiable positive function to be a Jacobian of some sense-preserving harmonic mapping. The relationship between a Jacobian of a harmonic mapping and the Schwarzian derivative of its dilatation is revealed. The structure of the set of harmonic mappings with a given Jacobian is described. The results are illustrated by examples. In conclusion, we consider an application of the main results of the article to the construction of variational formulas in classes of harmonic mappings with a given Jacobian.

Key words: *planar harmonic mappings, Jacobian, dilatation, Schwarzian derivative*

2020 Mathematical Subject Classification: *31A05*

1. Introduction. Recall [2] that the harmonic mapping $f(z)$ defined in a simply connected domain $D \subset \mathbb{C}$ can be represented in the form $f(z) = h(z) + \overline{g(z)}$, where $h(z)$, $g(z)$ are holomorphic in D functions. The functions $h(z)$ and $\overline{g(z)}$ are called holomorphic and antiholomorphic parts of the harmonic mapping $f(z)$, respectively. The Jacobian of a harmonic mapping $f(z)$ is given by the formula $J(z) = |h'(z)|^2 - |g'(z)|^2$ and is infinitely differentiable in D. By virtue of Lewy's theorem [2], the harmonic mapping $f(z)$ is sense-preserving in D iff $J(z) > 0$ in D. Throughout the paper, we deal with sense-preserving harmonic mappings defined on simply connected domains. The dilatation $\omega(z) = g'(z)/h'(z)$ of a sense-preserving mapping $f(z)$ is a holomorphic function with $|\omega(z)| < 1$ in D.

It is well known that the Jacobian $J(z) = |f'(z)|^2$ of a holomorphic in D function $f(z)$ is a non-negative function, such that $\ln J(z)$ is harmonic in D except of zeros of $J(z)$. Conversely, if we define in D a non-negative function $J(z) \in C^{\infty}(D)$, such that $\ln J(z)$ is harmonic away from the set where $J(z) = 0$, then, from the equality $J(z) = |f'(z)|^2$, we can, obviously, restore a class of holomorphic functions with the Jacobian $J(z)$. All functions of this class differ from each other in translation and rotation.

It is clear that several harmonic in D mappings can have the same Jacobian. In particular, if the function $J(z)$ is a Jacobian of the mapping $f(z) = h(z) + \overline{g(z)}$, then the mapping $\tilde{f}(z) = e^{i\alpha}h(z) + \overline{e^{i\beta}g(z)} + c$, $\alpha, \beta \in \mathbb{R}$, $c \in \mathbb{C}$, also has the Jacobian $J(z)$. At the same time, for example, the mappings $\hat{f}(z) = z\sqrt{J+c} + \overline{z}\sqrt{c}$, where the real constant $J > 0$ is fixed and the parameter $c > 0$, have different dilatations $\omega(z) = \sqrt{c}/\sqrt{J+c}$ and the same Jacobian J. Hence, the class of harmonic mappings $f(z)$ with the given Jacobian in the general case is not limited to translation and mutual rotation of the holomorphic and antiholomorphic parts of the harmonic mapping.

As a result, natural questions arise: which positive functions $J(z) \in C^{\infty}(D)$ are the Jacobians of harmonic mappings and how ambiguous is the class of sense-preserving harmonic in D mappings with a given Jacobian $J(z)$?

Note that the study of univalent harmonic mappings with dilatations of a given form, which is related in some sense, was considered, for example, by W. Hengartner and G. Schober [7]. In addition, one can notice the similarity between the problem of describing harmonic mappings with the given Jacobian and the well-known Keller conjecture [10, 12], included in 1998 in the list of 18 mathematical problems of the next century.

At the same time, the results obtained in this article differ significantly from the above works.

Recall also [3, 9], that the Schwarzian derivative of a locally univalent holomorphic function $f(z)$ defined in D is given by the formula

$$S[f,z] = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

is meromorphic in $\mathbb{D}$, and has the following well-known properties:

1. The Schwarzian derivative $S[f,z] \equiv 0$ in D iff $f(z) \equiv L(z)$, where $L(z)$ is a linear fractional function of the form

$$L(z) = \frac{a\,z+b}{c\,z+d}, \quad ad - bc \neq 0.$$

2. Let a composition $g \circ f(z)$ of holomorphic functions $f(z)$ and $g(z)$ be defined in some domain D. Then

$$S[g \circ f, z] = S\left[g, f(z)\right] \cdot \left(f'(z)\right)^2 + S\left[f, z\right].$$

As a consequence of these properties, we have

3. The Schwarzian derivatives of two holomorphic functions $f(z)$ and $g(z)$ coincide iff $g(z) = L \circ f(z)$, where L is some linear fractional transformation.

The Schwarzian derivative plays an important role in the complex analysis. For example, in 1949 Z. Nehari [11] used $S[f, z]$ to obtain a sufficient condition for univalence of holomorphic functions. The concept of the Schwarzian derivative has been repeatedly generalized, for example, to the case of harmonic mappings [1, 4, 8].

By the well-known Riemann mapping theorem, any two simply connected domains distinct from the complex plane $\mathbb{C}$ are conformally equivalent. And since the composition $f \circ g$, where f, g are harmonic and holomorphic in D functions, respectively, is harmonic in D, then it suffices to consider two cases: $D = \mathbb{C}$ and $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

If a harmonic mapping $f(z)$ is sense-preserving on $\mathbb{C}$, then, as is well known [2], the mapping $f(z)$ has the form $f(z) = h(z) + c_1 \cdot \overline{h(z)} + c_2$, where c_1,$c_2 \in \mathbb{C}$, $|c_1| < 1$ and $h(z)$ is holomorphic. Indeed, in this case a dilatation of $f(z)$ is a bounded holomorphic function. Hence, by virtue of Liouville's theorem, $\omega(z) \equiv$ const on $\mathbb{C}$. Thus, the case of sense-preserving on $\mathbb{C}$ harmonic mappings is quite simple and, as shown below, fits into the general picture.

The main case will be $D = \mathbb{D}$.

2. Main results. Let $f(z) = h(z) + \overline{g(z)}$ be a sense-preserving harmonic in $\mathbb{D}$ mapping. Then its Jacobian $J(z)$ and dilatation $\omega(z)$ have the form $J(z) = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$, respectively. Obviously, $J(z)$ and $\omega(z)$ are related by the ratio

$$J(z) = |h'(z)|^2(1 - |\omega(z)|^2). \tag{1}$$

The following theorem is a criteria for an infinitely differentiable positive function $J(z)$ to be a Jacobian of some sense-preserving harmonic mapping $f(z)$. This theorem also establishes an additional connection between its Jacobian and dilatation.

Theorem 1. *A positive function $J(z) \in C^\infty(\mathbb{D})$ is the Jacobian for some sense-preserving harmonic in $\mathbb{D}$ mapping $f(z) = h(z) + \overline{g(z)}$ iff there exists a holomorphic function $\omega(z)$ with $|\omega(z)| < 1$ in $\mathbb{D}$, such that*

$$(-\ln\, J(z))_{z\bar{z}} = \frac{|\omega'(z)|^2}{(1 - |\omega(z)|^2)^2}. \tag{2}$$

As a consequence, the Jacobian $J(z)$ and the dilatation $\omega(z)$ of any sense-preserving harmonic mapping are related by one of the following conditions:

1) *either function $\ln\, J(z)$ is harmonic in $\mathbb{D}$ (i. e., $(\ln\, J(z))_{z\bar{z}} \equiv 0$ in $\mathbb{D}$) and the function $\omega(z) \equiv \mathrm{const}$;*
2) *or the function $R(z) = \ln(-J^2(z) \cdot (\ln\, J(z))_{z\bar{z}})$ is defined and harmonic in $\mathbb{D} \setminus Z$, where Z is a set of isolated zeros of the function $(-\ln\, J(z))_{z\bar{z}}$. In this case, the function $Q(z) = 2\,(\ln(-\ln\, J(z))_{z\bar{z}})_{zz} - (\ln(-\ln\, J(z))_{z\bar{z}})_z^2$ is holomorphic in $\mathbb{D} \setminus Z$, and the functions $\omega(z)$ is a solution of the differential equation*

$$2S[\omega, z] = Q(z), \tag{3}$$

defined in $\mathbb{D} \setminus Z$.

In both cases, the function $J(z)$ satisfies the sharp inequality

$$(-\ln\, J(z))_{z\bar{z}} \leqslant \frac{1}{(1 - |z|^2)^2}. \tag{4}$$

Proof. *Necessity.* Let $f(z) = h(z) + \overline{g(z)}$ be a sense-preserving harmonic in $\mathbb{D}$ mapping. The equality (1) and the property of the modulus of a holomorphic function give:

$$(-\ln\, J(z))_{z\bar{z}} = -\left(\ln |h'(z)|^2 + \ln (1 - |\omega(z)|^2)\right)_{z\bar{z}} = \frac{|\omega'(z)|^2}{(1 - |\omega(z)|^2)^2},$$

i. e., there exists a holomorphic function $\omega(z)$ (which is the dilatation of the harmonic mapping $f(z)$) with $|\omega(z)| < 1$ in $\mathbb{D}$, such that the equality (2) holds everywhere in $\mathbb{D}$. Necessity is proved.

Sufficiency. Let a positive function $J(z) \in C^\infty(\mathbb{D})$ and suppose there exists a holomorphic function $\omega(z)$ with $|\omega(z)| < 1$ in $\mathbb{D}$, such that the equality (2) holds everywhere in $\mathbb{D}$. Let us prove that there exists a harmonic mapping $f(z)$ defined in $\mathbb{D}$ with the Jacobian $J(z)$.

To do this, consider the system

$$\begin{cases} |h'(z)|^2 - |g'(z)|^2 = J(z), \\ \dfrac{|g'(z)|}{|h'(z)|} = |\omega(z)|. \end{cases} \tag{5}$$

The solution of the system (5) has the form

$$|h'(z)| = \sqrt{\frac{J(z)}{1-|\omega(z)|^2}}, \qquad |g'(z)| = \sqrt{\frac{J(z)\cdot|\omega(z)|^2}{1-|\omega(z)|^2}} = |h'(z)|\cdot|\omega(z)|.$$

Since the function $\omega(z)$ satisfies the condition (2) in $\mathbb{D}$,

$$\left(\ln\,|h'(z)|^2\right)_{z\overline{z}} = \left(\ln\,\frac{J(z)}{1-|\omega(z)|^2}\right)_{z\overline{z}} \equiv 0.$$

So, $|h'(z)|$ is the modulus of some holomorphic function, and, as a consequence, the system (5) allows us to find the holomorphic in $\mathbb{D}$ functions $h'(z)$ and $g'(z)$ up to rotations. Integrating these functions in $\mathbb{D}$, we get $h(z)$ and $g(z)$ up to additive constants and, at the same time, a sense-preserving harmonic in $\mathbb{D}$ mapping $f(z) = h(z) + \overline{g(z)}$ with the given Jacobian $J(z)$. Sufficiency is proved.

Proof of the conditions 1) and 2). Let $f(z) = h(z) + \overline{g(z)}$ be a sense-preserving harmonic mapping defined in $\mathbb{D}$ with the Jacobian $J(z)$ and dilatation $\omega(z)$.

It is clear that only one of the following situations can be realized: either $\omega(z) \equiv \text{const} = c$, $|c| < 1$, or $\omega(z) \not\equiv const$ in $\mathbb{D}$ and all zeroes of $\omega'(z)$ are isolated.

1) By virtue of (2), equality $\omega(z) \equiv \text{const}$ is possible if and only if $(\ln\,J(z))_{z\overline{z}} \equiv 0$.

2) Let $\omega(z) \not\equiv \text{const}$ and, hence, $(-\ln\,J(z))_{z\overline{z}} \not\equiv 0$ in $\mathbb{D}$. From (2) it follows that the function $(-\ln\,J(z))_{z\overline{z}}$ is non-negative, i.e., the function $-\ln\,J(z)$ is subharmonic. Hence, the function $R(z) = \ln(-J^2(z)\cdot(\ln\,J(z))_{z\overline{z}})$ is defined in $\mathbb{D}\setminus Z$, where Z is a set of isolated zeros of the holomorphic function $\omega'(z)$.

To show that the function $R(z)$ is harmonic in $\mathbb{D}\setminus Z$, we rewrite (2) as

$$(-\ln\,J(z))_{z\overline{z}} = \frac{|\omega'(z)\cdot(h'(z))^2|^2}{J^2(z)}.$$

Therefore, $R(z) = \ln|\omega'(z)\cdot(h'(z))^2|^2$, so, $R(z)$ is harmonic in $\mathbb{D}\setminus Z$ as a real part of a holomorphic function.

Define the function $p(z) = (\ln(-\ln\, J(z))_{z\overline{z}})_z$. Then the function

$$Q(z) = 2\,(\ln(-\ln\, J(z))_{z\overline{z}})_{zz} - (\ln(-\ln\, J(z))_{z\overline{z}})_z^2 = 2p_z(z) - p^2(z).$$

Taking into account the equality (2), we have:

$$p(z) = (\ln(-\ln\, J(z))_{z\overline{z}})_z = \left(\ln\frac{|\omega'(z)|^2}{(1-|\omega(z)|^2)^2}\right)_z = \frac{\omega''(z)}{\omega'(z)} + 2\cdot\frac{\omega'(z)\cdot\overline{\omega(z)}}{1-|\omega(z)|^2},$$

$$p_z(z) = \frac{\omega'''(z)\cdot\omega'(z) - (\omega''(z))^2}{(\omega'(z))^2} + 2\cdot\frac{\omega''(z)\cdot\overline{\omega(z)}}{1-|\omega(z)|^2} + 2\cdot\left(\frac{\omega'(z)\cdot\overline{\omega(z)}}{1-|\omega(z)|^2}\right)^2.$$

After substitution we get

$$Q(z) = 2p_z(z) - p^2(z) = 2\left(\frac{\omega'''(z)}{\omega'(z)} - \frac{3}{2}\left(\frac{\omega''(z)}{\omega'(z)}\right)^2\right) = 2S[\omega, z].$$

Hence, the function $Q(z)$ is holomorphic at the points $z\in\mathbb{D}$, where $\omega'(z)\neq 0$, i. e., in $\mathbb{D}\setminus Z$, and the dilatation of the sense-preserving in $\mathbb{D}$ harmonic mapping $f(z)$ satisfies the differential equation (3).

Finally, since the dilatation $\omega(z)$ is holomorphic in $\mathbb{D}$ and $|\omega(z)| < 1$, then, by virtue of the Schwarz lemma [6] and (2), the inequality (4) characterizing a growth rate of the function $(-\ln\, J(z))_{z\overline{z}}$ holds in $\mathbb{D}$. The estimate is exact. It is attained, for example, on the function $f(z) = z + \overline{z}^2/2$, whose Jacobian has the form $J_f(z) = 1 - |z|^2$. $\square$

Remark. *Although the definition of the function Q in Theorem 1 does not require the existence of derivatives higher than the fourth order of the function J, in fact, since the function J must be the Jacobian of some harmonic mapping, it must belong to the class $C^\infty(\mathbb{D})$ as it stated in the condition of the theorem.*

The following result describes the structure of the family of all sense-preserving harmonic mappings defined in $\mathbb{D}$ with a given Jacobian $J(z)$.

Theorem 2. *Let $f_0(z) = h_0(z) + \overline{g_0(z)}$ be a sense-preserving harmonic in $\mathbb{D}$ mapping with dilatation $\omega_0(z) = g_0'(z)/h_0'(z)$ and Jacobian $J_0(z)$, $(\ln\, J_0(z))_{z\overline{z}} \not\equiv 0$ in $\mathbb{D}$. Then a harmonic in $\mathbb{D}$ mapping $f(z)$ has the Jacobian $J_0(z)$ iff its dilatation $\omega(z) = T\circ\omega_0(z)$, where*

$$T(w) = e^{i\alpha}\frac{w + w_0}{1 + \overline{w_0}\, w}, \quad w_0\in\mathbb{D}, \alpha\in\mathbb{R}.$$

Proof. Let the sense-preserving harmonic mapping $f(z) = h(z) + \overline{g(z)}$ have the dilatation $\omega(z) = g'(z)/h'(z)$. As has been marked above, the Schwarzian derivative is invariant under the linear fractional transformations L of the first argument. Hence, $S[\omega, z] \equiv S[\omega_0, z] \equiv 2Q(z)$ iff $\omega(z) = L \circ \omega_0(z)$. In view of the Theorem 1, dilatations of both mappings $f(z)$ and $f_0(z)$ with the same Jacobians should satisfy the equation (2). Therefore,

$$\frac{|\omega_0'(z)|^2}{(1-|\omega_0(z)|^2)^2} = \frac{|\,(L\circ\omega_0(z))'\,|^2}{(1-|L\circ\omega_0(z)|^2)^2}. \tag{6}$$

So, we have to prove that (6) is true iff $L = T$.

Sufficiency. If L is a linear fractional automorphism T of the disk $\mathbb{D}$, then the composition $T \circ \omega_0(z)$ satisfies the condition (2). Indeed, after substitution of $T \circ \omega_0(z)$ into the right-hand side of (6) we have:

$$\frac{\left|\left(\dfrac{\omega_0(z)+w_0}{1+\overline{w_0}\,\omega_0(z)}\right)'\right|^2}{\left(1-\left|\dfrac{\omega_0(z)+w_0}{1+\overline{w_0}\,\omega_0(z)}\right|^2\right)^2} = \frac{|\omega_0'(z)|^2\cdot(1-|w_0|^2)^2}{(|1+\overline{w_0}\,\omega_0(z)|^2-|\omega_0(z)+w_0|^2)^2} = \frac{|\omega_0'(z)|^2}{(1-|\omega_0(z)|^2)^2},$$

where $w_0 \in \mathbb{D}$.

Necessity. Let us show that the equality (6) is possible only if $L = T$.

Let the linear fractional transformation L be such that $L \circ \omega_0(\mathbb{D}) \subset \mathbb{D}$ and $L \circ \omega_0(z)$ satisfies the equality (6). In the general case

$$L(w) = \frac{aw+b}{cw+d}, \quad a,b,c,d \in \mathbb{C},\ ad-bc \neq 0.$$

For $a \neq 0$, the transformation L can be written as

$$L(w) = \frac{w+b}{cw+d}, \quad b,c,d \in \mathbb{C}.$$

Using the outer composition L with a linear fractional automorphism

$$T_0(w) = e^{i\alpha}\frac{w+w_0}{1+\overline{w_0}w}, \quad w_0 = -\frac{b}{d}$$

of the disk $\mathbb{D}$, we can get the following representation:

$$\widetilde{L}(w) = T_0 \circ L(w) = \frac{w}{\widetilde{c}w+\widetilde{d}}, \quad \widetilde{c},\widetilde{d} \in \mathbb{C},$$

where, without loss of generality, we can assume that $\widetilde{d} > 0$.

As mentioned above, $\widetilde{L} \circ \omega_0(z)$ satisfies the equality (6), i. e.,

$$\frac{\left|\left(\widetilde{L} \circ \omega_0(z)\right)'\right|}{1 - \left|\widetilde{L} \circ \omega_0(z)\right|^2} = \frac{\widetilde{d} \cdot |\omega_0'(z)|}{|\widetilde{c}\,\omega_0(z) + \widetilde{d}|^2 - |\omega_0(z)|^2} =$$

$$= \frac{\widetilde{d} \cdot |\omega_0'(z)|}{|\omega_0(z)|^2(|\widetilde{c}|^2 - 1) + \widetilde{d}^2 + 2 \cdot \widetilde{d} \cdot \mathtt{Re}(\widetilde{c}\,\omega_0(z))} \equiv \frac{|\omega_0'(z)|}{1 - |\omega_0(z)|^2}.$$

Since $\omega_0(z) \not\equiv \text{const}$ in $\mathbb{D}$, the last identity is equivalent to the following:

$$\frac{\widetilde{d}}{|\omega_0(z)|^2(|\widetilde{c}|^2 - 1) + \widetilde{d}^2 + 2 \cdot \widetilde{d} \cdot \mathtt{Re}(\widetilde{c}\,\omega_0(z))} \equiv \frac{1}{1 - |\omega_0(z)|^2}. \tag{7}$$

Consider some arc $\gamma \subset \mathbb{D}$ along which $|\omega_0(z)| \equiv \text{const} > 0$. Such arc obviously exists, since a non-constant holomorphic function is an open mapping. The right-hand side of the equality (7) is constant on γ. Then the left-hand side of (7) also must be constant. It is possible only if

$$\mathtt{Re}(\widetilde{c}\,\omega_0(z)) \equiv |\widetilde{c}| \cdot |\omega_0(z)| \cdot \cos(\arg\, \widetilde{c} + \arg\, \omega_0(z)) \equiv \text{const}.$$

The last identity implies that either $\arg \omega_0(z) \equiv \text{const}$ on γ, or $\widetilde{c} = 0$. If $\arg \omega_0(z) \equiv \text{const}$ on γ, then $|\omega_0(z)|$ and $\arg \omega_0(z)$ are constant on γ and, as a consequence, $\omega_0(z) \equiv \text{const}$ in $\mathbb{D}$. But this contradicts the condition. Then $\widetilde{c} = 0$ and the transformation $\widetilde{L}(w)$ has the form

$$\widetilde{L}(w) = \frac{w}{\widetilde{d}}.$$

Substitute $\widetilde{L} \circ \omega(z)$ into (6). We get:

$$\frac{\widetilde{d}}{\widetilde{d}^2 - |\omega(z)|^2} \equiv \frac{1}{1 - |\omega(z)|^2}.$$

This identity is obviously valid only in the case when $\widetilde{d} = 1$.

Summarizing everything above, we conclude that $\widetilde{L} = T_0 \circ L = E$, where E is the identity transformation. Whence we get that $L = T_0^{-1}$, i. e., L is a linear fractional automorphism of the disk $\mathbb{D}$.

The case $c \neq 0$ is treated similarly.

Thus, the dilatation $\omega(z)$ of any harmonic mapping $f(z)$ with the Jacobian $J_0(z)$ is defined up to a linear fractional automorphism T of the disk $\mathbb{D}$, and the class of sense-preserving mappings $f(z) = h(z) + \overline{g(z)}$ with the Jacobian $J_0(z)$ is described by the equalities

$$|h'(z)| = \sqrt{\frac{J_0(z)}{1 - |T \circ \omega_0(z)|^2}}, \qquad |g'(z)| = \sqrt{\frac{J_0(z) \cdot |T \circ \omega_0(z)|^2}{1 - |T \circ \omega_0(z)|^2}}.$$

The proof is completed. □

Theorem 2 implies that the set of harmonic mappings with a given Jacobian is not compact in the topology of locally uniform convergence in $\mathbb{D}$.

Also, as a consequence of the main results we get the way of restoring the class of sense-preserving harmonic mappings $f(z)$ of the unit disk $\mathbb{D}$ with the given Jacobian $J(z)$.

If a positive function $J(z) \in C^\infty(\mathbb{D})$ is such that $(\ln J(z))_{z\bar{z}} \equiv 0$ in $\mathbb{D}$, then the only holomorphic solutions of the equation (2) are the functions $\omega(z) \equiv \text{const} = c_0$. Since we are interested in sense-preserving harmonic mappings defined in $\mathbb{D}$, we consider $|c_0| < 1$. Substituting $\omega(z) = c_0$ into the system (5) and solving this system, we find the harmonic mapping $f(z)$ with the Jacobian $J(z)$.

In the specific case $c_0 = 0$, the second equation of the system (5) implies that $g(z) \equiv \text{const}$, i. e., the restored mapping $f(z)$ is holomorphic in $\mathbb{D}$.

If a positive function $J(z) \in C^\infty(\mathbb{D})$ satisfies the case 2) of Theorem 1, then, as shown above, the differential equation (3) is defined in $\mathbb{D} \setminus Z$. Among the solutions of this differential equation, it suffices to find at least one function $\omega(z)$ (if it exists), such that $|\omega(z)| < 1$ and the equality (2) is satisfied in $\mathbb{D}$. Substituting $J(z)$ and $\omega(z)$ into the system (5), we can find one of the sense-preserving harmonic mappings of the disk $\mathbb{D}$ with the given Jacobian $J(z)$. All other members of this family can be found by substituting into the system (5) of functions $J(z)$ and $T \circ \omega(z)$, where T is a linear fractional automorphism of the disk $\mathbb{D}$.

3. Examples. Let us illustrate the main results.

Example 4. Consider separately the trivial case, when the Jacobian of a harmonic mapping satisfies the condition $(\ln J(z))_{z\bar{z}} \equiv 0$ in an arbitrary domain D. By virtue of the equality (2), we have $\omega(z) \equiv \text{const} = c$,

$|c| < 1$, in $\mathbb{D}$. Hence, $f(z) = h(z) + \overline{c \cdot h(z)} + c_1$, where $h(z)$ is a holomorphic function.

The reverse is also true. If the harmonic mapping has the mentioned above form, then

$$(\ln J(z))_{z\overline{z}} = (\ln(|h'(z)|^2 - |c \cdot h'(z)|^2))_{z\overline{z}} = (\ln |h'(z)|^2)_{z\overline{z}} + (\ln(1-|c|^2))_{z\overline{z}} \equiv 0.$$

In particular, if $D = \mathbb{C}$, then, as noted above, a sense-preserving harmonic mapping $f(z) = h(z) + c \cdot \overline{h(z)} + c_1$, and, as a consequence, $(\ln J(z))_{z\overline{z}} \equiv 0$ on $\mathbb{C}$.

Example 5. Consider the function $J(z) = z + \overline{z} + 2$.

This function satisfies the inequality (4). Indeed, the inequality

$$(-\ln J(z))_{z\overline{z}} = \frac{1}{(z + \overline{z} + 2)^2} \leqslant \frac{1}{(1 - |z|^2)^2}$$

in the disk $\mathbb{D}$ is equivalent to the inequality $|z - 1| \leqslant 2$, validity of which in $\mathbb{D}$ is obvious.

The function $R(z) = \ln(-J^2(z) \cdot (\ln J(z))_{z\overline{z}}) \equiv 0$ is harmonic in $\mathbb{D}$, and the function $p(z) = (\ln(-\ln J(z))_{z\overline{z}})_z$ has the form $p(z) = -2\,(z + \overline{z} + 2)^{-1}$. Holomorphic in $\mathbb{D}$ function $Q(z) = 2p_z(z) - p^2(z) \equiv 0$. The only solutions of the equation $2S[\omega, z] = Q(z)$ are

$$\omega(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

In view of the condition $|\omega(z)| < 1$ in $\mathbb{D}$, only two cases are possible:

a) the closure of the disk $\omega(\mathbb{D})$ belongs to the open unit disk. This case is satisfied by functions $\omega(z)$ of the form

$$\omega(z) = re^{i\alpha}\frac{z + z_0}{1 + \overline{z_0}z} + w_0,\ \alpha \in \mathbb{R}, \quad z_0 \in \mathbb{D},\ |w_0| < 1,\ r \in (0; 1 - |w_0|);$$

b) the boundary circle $\omega(\partial\mathbb{D})$ touches the unit circle internally or coincides with it. This case is satisfied by functions $\omega(z)$ of the form

$$\omega(z) = re^{i\alpha}\frac{z + z_0}{1 + \overline{z_0}z} + (1 - r)e^{i\beta}, \quad \alpha, \beta \in \mathbb{R},\ z_0 \in \mathbb{D},\ r \in (0; 1].$$

The condition (2) for given $J(z) = z + \overline{z} + 2$ takes the form

$$\frac{1}{(z + \overline{z} + 2)^2} = \frac{|\omega'(z)|^2}{(1 - |\omega(z)|^2)^2}. \tag{8}$$

Note that the expression in the left-hand side of the equality (8) approaches to infinity as $z \to -1$. Hence, the expression in the right-hand side of (8) also must have this property.

Consider the functions $\omega(z)$ from the case a). For such $\omega(z)$, the right-hand side of (8) is bounded in $\mathbb{D}$, since $|\omega(z)| < 1 - |w_0| + r$ and the function $\omega'(z)$ is bounded in $\mathbb{D}$. I. e., in the case a) there are no suitable functions $\omega(z)$.

Consider now the functions $\omega(z)$ from the case b). Let $\omega(z)$ satisfy the condition (8).

If $r = 1$, $\alpha, \beta \in \mathbb{R}$, $z_0 \in \mathbb{D}$, then the right-hand side of (8) approaches to infinity as z approaches to any point on the unit circle. At the same time, the left-hand side of the equality (8) grows infinitely only if $z \to -1$. Hence, for $r = 1$ there are no necessary functions $\omega(z)$.

Let $r \in (0{,}1)$ now and $z_0 \in \mathbb{D}$. Then, using the appropriate rotation of the function $\omega(z)$, we can bring it to the form

$$\omega_1(z) = re^{i\alpha_1}\frac{z + z_0}{1 + \overline{z_0}z} + 1 - r,$$

where $\alpha_1 = -\arg\left((-1 + z_0)\left(1 - \overline{z_0}\right)^{-1}\right)$, because left-hand and right-hand sides of (8) must approach to infinity as $z \to -1$. Thus, under the action of the function $\omega_1(z)$, the unit disk is mapped onto the disk D_r of radius r, tangent to the unit circle internally at the point $z = 1$.

By virtue of Theorem 2 the equality (8) also holds for $\omega_1(z)$. Let us apply the transformation

$$T(w) = \frac{w + b}{1 + bw}, \quad b = \frac{r - R}{R - 2Rr + r}, \quad R \in (0;1)\,,$$

to the function $\omega_1(z)$. Direct calculations show that $b \in (-1;1)$, i. e., $T(w)$ is an automorphism of the unit disk $\mathbb{D}$, such that $T(1) = 1$ and $T(1 - 2r) = 1 - 2R$. So, the function $T(w)$ maps the disk D_r onto the disk D_R of arbitrary radius $R \in (0;1)$, also tangent to the unit circle at the point $w = 1$. In view of uniqueness of conformal mapping with the given normalizations, we have:

$$T \circ \omega_1(z) = Re^{i\theta}\frac{z + c}{1 + \bar{c}z} + 1 - R$$

with $c \in \mathbb{D}$ and arbitrary $R \in (0;1)$.

In view of Theorem 2, the equality (8) also holds for $T \circ \omega_1(z)$:

$$\frac{1}{(z+\overline{z}+2)^2} = \frac{R^2(1-|c|^2)^2}{\left(|1+\overline{c}z|^2 - |Re^{i\theta}(z+c)+(1-R)(1+\overline{c}z)|^2\right)^2}.$$

For $z=0$ and for all $R \in (0;1)$, we have:

$$\frac{1}{4} = \frac{R^2\left(1-|c|^2\right)^2}{\left(1-|Re^{i\theta}c+1-R|^2\right)^2}.$$

Taking the limit as $R \to 1-$ at the right-hand side of this equality, we get the contradiction $1/4 = 1$.

Thus, there are no sense-preserving harmonic in $\mathbb{D}$ mappings $f(z) = h(z) + \overline{g(z)}$ with the given Jacobian $J(z) = z + \overline{z} + 2$.

Example 6. Let $J(z) = 1 - |z|^{2n}$, $n \in \mathbb{N}$.

The function $R(z) = \ln\left(-J^2(z) \cdot (\ln\, J)_{z\overline{z}}\right) = 2\ln\, n + (n-1)\ln|z|^2$ is harmonic in $\mathbb{D} \setminus \{0\}$. The corresponding function $Q(z)$ has the form

$$Q(z) = \frac{1-n^2}{z^2}.$$

It is checked directly that $\omega_0(z) = z^n$ is one of the solutions of the differential equation $2S[\omega, z] = (1-n^2)\cdot z^{-2}$. More than that, the function z^n satisfies the conditions $|\omega_0(z)| < 1$ and (2) in $\mathbb{D}$:

$$(-\ln\, J(z))_{z\overline{z}} = \frac{|nz^{n-1}|^2}{(1-|z|^{2n})^2} = \frac{|\omega_0'(z)|^2}{(1-|\omega_0(z)|^2)^2}.$$

So, $\omega_0(z) = z^n$ is one of the admissible dilatations of harmonic in $\mathbb{D}$ mappings with the given Jacobian $J(z) = 1 - |z|^{2n}$. Therefore, Theorem 2 claims that any admissible dilatation has the form

$$\omega(z) = T \circ \omega_0(z) = e^{i\alpha}\frac{z^n + z_0}{1+\overline{z_0}\, z^n}, \quad z_0 \in \mathbb{D}, \alpha \in \mathbb{R}.$$

Thus,

$$|h'(z)|^2 = \frac{J(z)}{1-\left|\dfrac{z^n+z_0}{1+\overline{z_0}\, z^n}\right|^2} = \frac{1-|z|^{2n}}{1-\left|\dfrac{z^n+z_0}{1+\overline{z_0}\, z^n}\right|^2} = \frac{|1+\overline{z_0}\, z^n|^2}{1-|z_0|^2},$$

$$|g'(z)|^2 = |h'(z)|^2 \cdot |\omega(z)|^2 = \frac{|z^n + z_0|^2}{1 - |z_0|^2}.$$

Whence, we get

$$f(z) = \frac{e^{i\alpha}}{\sqrt{1-|z_0|^2}}\left(z + \frac{\overline{z_0}}{n+1}z^{n+1}\right) + \frac{e^{-i\beta}}{\sqrt{1-|z_0|^2}}\overline{\left(\frac{1}{n+1}z^{n+1} + z_0 z\right)} + C,$$

where $z_0 \in \mathbb{D}$, $\alpha, \beta \in \mathbb{R}$, $C \in \mathbb{C}$.

4. Applications. The results proved in Section 2 can be used, for example, to construct variational formulas in the classes of harmonic mappings with a given Jacobian in solving extremal problems of the theory of harmonic mappings. The similarity of these problems with the classical problems of the theory of conformal mappings and, at the same time, their originality, as well as significant differences with the holomorphic case, lead to the fact that extremal problems in the theory of harmonic mappings remain relevant and attractive in the modern geometric function theory. This is evidenced by a significant number of publications devoted to this topic [2, 5, 8].

Let $\mathcal{F}(J)$ be a set of sense-preserving in the unit disk $\mathbb{D}$ harmonic mappings $f(z) = h(z) + \overline{g(z)}$ with the common Jacobian $J(z)$. Consider the continuous functional

$$L(f,r) = \int\limits_{|z|=r} (|h'(z)| - |g'(z)|)|dz| = \int\limits_{|z|=r} \sqrt{J(z)}\sqrt{\frac{1-|\omega(z)|}{1+|\omega(z)|}}|dz|, \quad r \in (0,1). \tag{9}$$

This functional is obviously a rather rough lower bound for the length $\int_{|z|=r} |df(z)|$ of the circle image under the mapping $f(z)$.

We can further assume that the function $\ln J(z)$ is not harmonic, since otherwise the functional $L(f,r)$ would be constant on the class $\mathcal{F}(J)$. Also, we will consider harmonic mappings whose dilatations differ by rotation as coinciding, because the values of the functional $L(f,r)$ for such functions are equal. The question of attaining the minimal or maximal values of the functional $L(f,r)$ for $f(z) \in \mathcal{F}(J)$ in the general case can be quite complicated, since, as shown above, the class $\mathcal{F}(J)$ is not compact. Nevertheless, Theorem 2 allows us to obtain a necessary condition for an extremum $L(f,r)$ on the class $\mathcal{F}(J)$.

Theorem 3. *Let the mapping $f_0(z) = h_0(z)+\overline{g_0(z)}$ be an extremal (strict minimum or maximum) point of the functional $L(f,r)$ on the class $\mathcal{F}(J)$ and have the dilatation $\omega_0(z)$ (up to rotation). Then*

$$\int\limits_{|z|=r} \sqrt{J(z)}\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}e^{i\arg\,\omega_0(z)}|dz| = 0. \tag{10}$$

Proof. Let a sense-preserving mapping $f_0(z) = h_0(z) + \overline{g_0(z)}$ be, for definiteness, a maximum point of $L(f,r)$ on the class $\mathcal{F}(J)$ and have the dilatation $\omega_0(z)$. By virtue of Theorem 2, any other function $f(z) \in \mathcal{F}(J)$ has a dilatation $\omega(z)$, such that

$$|\omega(z)| = \left|\frac{\omega_0(z)+a}{1+\overline{a}\,\omega_0(z)}\right|, \quad 0<|a|<1,$$

and $L(f,r) < L(f_0,r)$. The nonconstant holomorphic function $\omega_0(z)$ can have only isolated zeros in $\mathbb{D}$. We can assume that $\omega_0(z) \neq 0$ on $|z| = r$. Otherwise we will exclude sufficiently small arcs of $|z| = r$ in (9), such that $|\omega_0(z)| \geqslant c > 0$ on the remaining part γ of the circle and the inequality

$$\int\limits_{\gamma} \left(|h_0'(z)| - |g_0'(z)|\right)|dz| > L\,(f,r)$$

still be true.

So, $\omega_0(z) \neq 0$ on $|z| = r$. Then, for small values of $|a| = \varepsilon > 0$, the following asymptotic formulas are true:

$$\begin{aligned} |\omega(z)|^2 = |\omega_0(z)+a|^2\cdot|1-\overline{a}\omega_0(z)+o\,(a,z)|^2 = \\ = \left|\omega_0(z)+a-\overline{a}\omega_0^2(z)+o(a,z)\right|^2 = \\ = |\omega_0(z)|^2 + 2\varepsilon\,|\omega_0(z)|\left(1-|\omega_0(z)|^2\right)\cos\left(\arg\,\omega_0(z)-\arg\,a\right)+o(\varepsilon,z), \end{aligned}$$

$$|\omega(z)| = |\omega_0(z)| + \varepsilon(1-|\omega_0(z)|^2)\cos(\arg\,\omega_0(z)-\arg\,a)+o(\varepsilon,z),$$

$$\sqrt{\frac{1-|\omega(z)|}{1+|\omega(z)|}} = \sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}\left(1-\varepsilon\cos(\arg\,\omega_0(z)-\arg\,a)+o(\varepsilon,z)\right),$$

where $o(\varepsilon,z)$ are uniformly small on the circle $|z| = r$. Substituting the last asymptotics into (9), we obtain the expression for the variation of the functional $L(f_0,r)$:

$$\delta L(f_0,r)=L(f,r)-L(f_0,r)=\int\limits_{|z|=r}\sqrt{J(z)}\left(\sqrt{\frac{1-|\omega(z)|}{1+|\omega(z)|}}-\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}\right)|dz|=$$

$$=-\varepsilon\int\limits_{|z|=r}\sqrt{J(z)}\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}\cos(\arg\,\omega_0(z)-\arg\,a)|dz|+o(\varepsilon)=$$

$$=-\varepsilon\cdot\mathrm{Re}\,e^{-i\arg\,a}\int\limits_{|z|=r}\sqrt{J(z)}\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}e^{i\arg\,\omega_0(z)}|dz|+o(\varepsilon)$$

with some small constants $o(\varepsilon)$. Since the mapping $f_0(z)$ is a maximum point of the functional $L(f,r)$, the inequality $\delta L(f_0,r)\leqslant 0$ is valid for any values of $\arg\,a$ and all small $\varepsilon=|a|$. Therefore, the main part of the variation $\delta L(f_0,r)$ must be equal to zero:

$$-\varepsilon\cdot\mathrm{Re}\,e^{-i\arg\,a}\int\limits_{|z|=r}\sqrt{J(z)}\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}e^{i\arg\,\omega_0(z)}|dz|=0.$$

The condition (10) follows immediately in view of arbitrariness of $\arg\,a$. $\square$

For example, if $J_0(z)=1-|z|^2$, then, as shown in Example 3, mappings of the class $\mathcal{F}(J_0)$ consists of functions with dilatations $\omega(z)=$ $=e^{i\alpha}\,(z-c)\,(1-\bar{c}\,z)^{-1}$, $|c|<1$. Numerical estimates show that among them only $\omega_0(z)=e^{i\alpha}z$ satisfy the condition (10):

$$\int\limits_{|z|=r}\sqrt{J(z)}\sqrt{\frac{1-|\omega_0(z)|}{1+|\omega_0(z)|}}e^{i\arg\,\omega_0(z)}|dz|=(1-r)re^{i\alpha}\int\limits_0^{2\pi}e^{it}dt=0.$$

The functional $L(f,r)$ reaches its maximal value on the mappings $f_0(z)=$ $=z+\frac{1}{2}\overline{e^{i\alpha}z^2}$ with dilatations $\omega_0(z)=e^{i\alpha}z$ in the class $\mathcal{F}(J_0)$.

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