DOI: 10.15393/j3.art.2024.14330

UDC 517.544

Yu. I. Krotova

INTEGRABILITY OF q-BESSEL FOURIER TRANSFORMS WITH GOGOLADZE – MESKHIA TYPE WEIGHTS

Abstract. In the paper, we consider the q-integrability of functions $\lambda(t)|\mathcal{F}_{q,\nu}(f)(t)|^r$, where $\lambda(t)$ is a Gogoladze-Meskhia-Moricz type weight and $\mathcal{F}_{q,\nu}(f)(t)$ is the q-Bessel Fourier transforms of a function f from generalized integral Lipschitz classes. There are some corollaries for power type and constant weights, which are analogues of classical results of Titchmarsh et al. Also, a q-analogue of the famous Herz theorem is proved.

Key words: *q-Bessel Fourier transform, q-Bessel translation, modulus of smoothness, weights of Gogoladze – Meskhia type, q-Besov space*

2020 Mathematical Subject Classification: *44A15, 47A10*

1. Introduction. Let $f\colon \mathbb{R} \to \mathbb{C}$ be an integrable function in Lebesgue's sense over $\mathbb{R}$ ($f \in L^1(\mathbb{R})$). Then the Fourier transform of f is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} f(t) e^{-itx}\, dt, \quad x \in \mathbb{R}.$$

In the case $1 < p \leqslant 2$, we define $\widehat{f}(x)$ as a limit in $L^q(\mathbb{R})$, $1/p + 1/q = 1$, by

$$\widehat{f}(x) = (L^q) \lim_{\substack{a\to-\infty,\\ b\to+\infty}} (2\pi)^{-1/2} \int\limits_a^b f(x) e^{-itx}\, dt.$$

In particular, $\widehat{f} \in L^q(\mathbb{R})$ and the following Hausdorff–Young type inequality proved by Titchmarsh (see [16, Ch. IV, Theorem 74])

$$\|\widehat{f}\|_q \leqslant C\|f\|_p := C\Big(\int\limits_{\mathbb{R}} |f(t)|^p\, dt\Big)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leqslant 2, \qquad (1)$$

holds. For $p = 2$ the inequality in (1) is substituted by the Plancherel equality. More about these results can be found in [16, Ch. III and IV] or [3, Ch. 5].

For $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, we consider the modulus of smoothness of order $k \in \mathbb{N}$:

$$\omega_k(t, \delta)_p = \sup_{0 \leqslant h \leqslant \delta} \|\mathring{\Delta}_h^k f\|_p, \quad \mathring{\Delta}_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k - 2j)h/2).$$

The following result of Titchmarsh is well known (see [16, Ch. 4, Theorem 84]):

Theorem 1. *Let $1 < p \leqslant 2$, $0 < \alpha \leqslant 1$, $f \in Lip(\alpha, p)$. Then $\widehat{f}(t) \in L^\beta(\mathbb{R})$ for all β satisfying the inequality*

$$\frac{p}{p + ap - 1} < \beta \leqslant q = \frac{p}{p - 1}.$$

Unfortunately, in [16] and in many papers where Theorem 1 is presented there is no information that this theorem is an analogue of Szasz's results for trigonometric series (see, e.g., [15] and [12]– [14] in literature from this paper).

We will write that a non-negative measurable function $\lambda(t) \in L^1_{loc}(\mathbb{R}_+)$ belongs to the class A_γ, $\gamma \geqslant 1$, if there exists $C(\gamma) \geqslant 1$, such that

$$\Big(\int_{2^i}^{2^{i+1}} \lambda^\gamma(t) dt \Big)^{1/\gamma} \leqslant C(\gamma) 2^{i(1-\gamma)/\gamma} \int_{2^{i-1}}^{2^i} \lambda(t) dt, \quad i \in \mathbb{Z}. \tag{2}$$

It is clear that a measurable function $\lambda(t) \geqslant 0$ with the property

$$\sup\{\lambda(t) \colon 2^i \leqslant t < 2^{i+1}\} \leqslant c \inf\{\lambda(t) \colon 2^{i-1} \leqslant t < 2^i\}, \quad i \in \mathbb{Z}.$$

is contained in all classes A_γ, $\gamma \geqslant 1$. From the last assertion, we deduce that $\lambda(t) = t^\alpha$, $\alpha \in \mathbb{R}$, belongs to all classes A_γ, $\gamma \geqslant 1$. Further we assume that $\lambda(t) = \lambda(-t)$ for $t > 0$.

An analogue of (2) for sequences was introduced by Gogoladze and Meskhia [9]. The condition (2) was suggested by Móricz [13] who proved the following result:

Theorem 2. *Let $1 < p \leqslant 2$ and $f \in L^p(\mathbb{R})$. If $1/p + 1/q = 1$, $0 < r < q$, and $\lambda \in A_{p/(p-rp+r)}$, then*

$$\int\limits_{|t|\geqslant 2} \lambda(t)|\widehat{f}(t)|^r dt \leqslant \int\limits_1^\infty \lambda(t)t^{-r/q}\omega^r(f,\,\pi/t)_p\,dt.$$

The aim of the present paper is to obtain an analogue and a generalization of Theorem 2 for the q-Bessel Fourier transform. Some analogues of Theorem 1 for Fourier-Bessel (or Hankel) transform proved by Platonov [14]. An analogue of Theorem 1 for q-Fourier-Dunkl transforms can be found in [5]. Analogues and extensions of Theorem 2 for Fourier-Dunkl transforms are proved by Volosivets [17], while for the first Hankel-Clifford transform see [18]. The main result of the present paper and its corollaries is similar to that of [17] and [18], but the subject of the present paper is discrete and methods are different from used in the cited papers. Also, we obtain an analogue of the famous Herz theorem (see Corollary 4).

2. Definitions and lemmas. Let $0 < q < 1$, $\nu > -1$, and $\mathbb{R}_q^+ = \{q^n : n \in \mathbb{Z}\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, set

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

Let us introduce the q-integral of Jackson for f defined on $\mathbb{R}_q^+$ on intervals from 0 to $a \in \mathbb{R}_q^+$ an from 0 to ∞, as follows:

$$\int\limits_0^a f(x)\,d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n; \qquad \int\limits_0^\infty f(x)\,d_qx = (1-q)\sum_{n\in\mathbb{Z}} q^n f(q^n).$$

Then, for $0 < a < b$, $a, b \in \mathbb{R}_q^+$, set

$$\int\limits_a^b f(x)\,d_qx = \int\limits_0^b f(x)\,d_qx - \int\limits_0^a f(x)\,d_qx.$$

For such integral, there is the following simple change of variables formula:

$$\int\limits_a^b h(x/r)x^{2\nu+1}d_qx = r^{2\nu+2}\int\limits_{a/r}^{b/r} h(t)t^{2\nu+1}d_qt, \quad r \in \mathbb{R}_q^+, \quad \nu > -1, \tag{3}$$

(see [1]). A more general variant is in [11, (19.14)].

The third Jackson q-Bessel function J_ν (also called Hahn-Exton q-Bessel function) is defined by

$$J_\nu(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1},q)_n (q,q)_n} x^{2n}.$$

We consider also its normalized form:

$$j_\nu(x;q) = \frac{(q,q)_\infty}{(q^{n+1},q)_\infty} x^{-\nu} J_\nu(x;q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1};q)_n (q;q)_n} x^{2n}.$$

These functions satisfy the orthogonality condition:

$$c_{q,\nu}^2 \int_0^\infty j_\nu(q^n x;q^2) j_\nu(q^m x;q^2) x^{2\nu+1}\, d_q x = \frac{q^{-2n(\nu+1)}}{1-q} \delta_{nm},$$

where δ_{nm} is the Kronecker symbol,

$$c_{q,\nu} = ((1-q)(q^2;q^2)_\infty)^{-1} (q^{2(\nu+1)};q^2)_\infty$$

(see [12]). Further we write $d\mu_{q,\nu}(x)$ instead of $x^{2\nu+1}\, d_q x$.

Let $\Delta_{q,\nu} f(x)$ be the q-Bessel operator defined by

$$\Delta_{q,\nu} f(x) = x^{-2}[f(q^{-1}x) - (1+q^{2\nu}) f(x) + q^{2\nu} f(qx)].$$

The function $j_\nu(\lambda x;q)$ is a solution of the following difference equation: $\Delta_{q,\nu} f(x) = -\lambda^2 f(x)$.

For $1 \leqslant p < \infty$, denote by $L^p_{q,\nu}$ the space of all real-valued functions f defined on $\mathbb{R}_q^+$ with finite norm

$$\|f\|_{p,q,\nu} = \Big(\int_0^\infty |f(x)|^p \, d\mu_{q,\nu}(x) \Big)^{1/p}.$$

If χ_E is the indicator of a set E and $f(x)\chi_E(x) \in L^p_{q,\nu}$, then we write $f \in L^p_{q,\nu}(E)$. The space $L^\infty_{q,\nu}$ consists of all bounded on $\mathbb{R}_q^+$ functions and is supplied by the usual sup-norm.

Define the q-Bessel Fourier transform $\mathcal{F}_{q,\nu}(f)$ for $f \in L^p_{q,\nu}$, $p \geqslant 1$, by

$$\mathcal{F}_{q,\nu}(f)(x) = c_{q,\nu} \int_0^\infty f(t) j_\nu(xt,q^2)\, d\mu_{q,\nu}(t).$$

Also, the q-Bessel translation operator is introduced by

$$T^{\nu}_{q,x}(f)(y) = c_{q,\nu} \int_0^{\infty} \mathcal{F}_{q,\nu}(f)(t) j_v(xt, q^2) j_v(yt, q^2)\, d\mu_{q,\nu}(t).$$

It is known that for $f \in L^1_{q,\nu}$ the equality

$$\int_0^{\infty} T^{\nu}_{q,x}(f)(y)\, d_{q,\nu}(y) = \int_0^{\infty} f(y)\, d_{q,\nu}(y) \tag{4}$$

holds (see [6, Proposition 5.2]). In [8], the problem of positivity of the operator $T^{\nu}_{q,x}$ was discussed by Fitouhi and Dhaouadi. If from $f \geqslant 0$ on $\mathbb{R}^+_q$ it follows that $T^{\nu}_{q,q^n} f \geqslant 0$ for all $n \in \mathbb{Z}$, then $T^{\nu}_{q,x}$ is called positive. For $\nu \geqslant 0$ and all $q \in (0,1)$, operators T^{ν}_{q,q^n}, $n \in \mathbb{Z}$, are positive, while for $-1 < \nu < 0$ the situation is more complicated. By virtue of this fact, we consider $\nu \geqslant 0$ further.

In [8], it is proved that $\|T^{\nu}_{q,x} f\|_{1,q,\nu} \leqslant \|f\|_{1,q,\nu}$ for a positive operator $T^{\nu}_{q,x}$ and $f \in L^1_{q,\nu}$. On the other hand, Dhaouadi [7] proved that

$$\|T^{\nu}_{q,x} f\|_{\infty,q,\nu} \leqslant \|f\|_{\infty,q,\nu}, \quad f \in L^{\infty}_{q,\nu},$$

and, by the Riesz-Thorin interpolation theorem, concluded that (see [7, Theorem 4])

$$\|T^{\nu}_{q,x} f\|_{p,q,\nu} \leqslant \|f\|_{p,q,\nu}, \quad 1 \leqslant p \leqslant \infty, \quad f \in L^p_{q,\nu}. \tag{5}$$

Note that

$$|j_{\nu}(x, q^2)| \leqslant 1, \quad x \in \mathbb{R}^+_q, \quad \nu \geqslant 0, \tag{6}$$

(see [7, Remark 1]). In [1] it is proved that the inequality (6) is strong for $x \in \mathbb{R}^+_q$. From (6), the inequality

$$\|\mathcal{F}_{q,\nu}(f)\|_{\infty,q,\nu} \leqslant c_{q,\nu} \|f\|_{1,q,\nu} \tag{7}$$

easily follows for $f \in L^1_{q,\nu}$.

For $\alpha > 0$, $m > 0$, $\theta \geqslant 1$, and $p \geqslant 1$, consider the q-Besov space $B^{\alpha,\theta,m}_{p,q,\nu}$ consisting of all $f \in L^p_{q,\nu}$, such that

$$\|f\|_B = \|f\|_{p,q,\nu} + \Big(\sum_{k\in\mathbb{Z}} (q^{-k\alpha} \omega_m(f, q^k)_{p,q,\nu})^{\theta}\Big)^{1/\theta} < \infty.$$

Let $M_k = [(1/q)^k, (1/q)^{k+1}]$. The Herz space $K(\alpha, \theta, p, q, \nu)$ contains all function f on $\mathbb{R}_q^+$, such that $f X_{M_k} \in L^p_{q,\nu}$ for all $k \in \mathbb{Z}$ and

$$\|f\|_K = \|f\|_{K(\alpha,\theta,p,q,\nu)} = \Big(\sum_{k\in\mathbb{Z}} \|q^{-\alpha k} f X_{M_k}\|^\theta_{p,q,\nu}\Big)^{1/\theta} < \infty.$$

Here X_E is the indicator of a set E.

Lemma 1.

(i) *Let* $f \in L^p_{q,\nu}$, $p \geqslant 1$. *Then* $\mathcal{F}^2_{q,\nu}(f)(x) = \mathcal{F}_{q,\nu}(\mathcal{F}_{q,\nu} f)(x) = f(x)$, $x \in \mathbb{R}_q^+$;

(ii) *If* $f \in L^p_{q,\nu}$, *then* $\mathcal{F}_{q,\nu}(f) \in L^2_{q,\nu}$ and $\|F_{q,\nu}(f)\|_{2,q,\nu} = \|f\|_{2,q,\nu}$;

(iii) *For any* $f \in L^p_{q,\nu}$, $p = 1, 2$, *the equality holds:*

$$\mathcal{F}_{q,\nu}(T^\nu_{q,x} f)(y) = j_\nu(yx, q^2)\mathcal{F}_{q,\nu}(f)(y), \quad y, x \in \mathbb{R}_q^+.$$

Proof. The statement (i) of Lemma 1 is proved in [6, Theorem 3.2], while (ii) is established in [7, Theorem 3]. The part (iii) is proved in [1] in the case $p = 2$, but in other cases the proof is the same. □

From (7) and Lemma 1 (ii) by Riesz-Thorin, a theorem follows:

Lemma 2. *Let* $1 \leqslant p \leqslant 2$, $1/p + 1/p' = 1$, $f \in L^p_{q,\nu}$. *Then* $\mathcal{F}_{q,\nu}(f) \in L^{p'}_{q,\nu}$ *and*

$$\|\mathcal{F}_{q,\nu}(f)\|_{p',q,\nu} \leqslant C\|f\|_{p,q,\nu}. \tag{8}$$

Since in (5) the constant in the right-hand side is equal to 1, for $m > 0$ the difference of order m with step h may be defined by

$$\Delta^m_{q,\nu,h} = (I - T^\nu_{q,h})^m = \sum_{j=0}^\infty (-1)^j \binom{m}{j} (T^\nu_{q,h})^j,$$

where I is an identical operator and

$$\binom{m}{j} = \frac{m(m-1)\dots(m-j+1)}{j!}, \quad j \in \mathbb{N}, \quad \binom{m}{0} = 1.$$

It is known that $\sum_{j=0}^\infty |\binom{m}{j}| < \infty$ for $m > 0$ (see, e.g., [4]), therefore, by (5) for $f \in L^p_{q,\nu}$ one has $\Delta^m_{q,\nu,h} f \in L^p_{q,\nu}$ and

$$\|\Delta^m_{q,\nu,h} f\|_{p,q,\nu} \leqslant C(m)\|f\|_{p,q,\nu} \tag{9}$$

From Lemma 1 (iii), it follows that for $f \in L^p_{q,\nu}$, $p = 1, 2$,

$$\mathcal{F}_{q,\nu}(\Delta^m_{q,\nu,h} f)(x) = (1 - j_\nu(yx, q^2))^m \mathcal{F}_{q,\nu}(f)(x), \quad x \in \mathbb{R}^+_q. \tag{10}$$

The next Lemma can be found in [1]:

Lemma 3. *There exist there exist $\alpha, \beta, \gamma > 0$, such that*

$$\alpha \leqslant |j_\nu(t, q^2) - 1|, \quad t \in \mathbb{R}^+_q \cap [1, +\infty),$$

$$\eta t^2 \leqslant |j_\nu(t, q^2) - 1| \leqslant \beta t^2, \quad t \in \mathbb{R}^+_q \cap (0, 1].$$

We define the modulus of smootheness of order $m > 0$ for $f \in L^p_{q,\nu}$ by

$$\omega_m(f, \delta)_{p,q,\nu} = \sup_{0<h\leqslant\delta} \|\Delta^m_{q,\nu,h} f\|_{p,q,\nu}$$

We will write $\lambda(t) \in A_{\gamma,q,\nu}$, $\gamma \geqslant 1$, if for $i \in \mathbb{Z}$ the inequality

$$\Big(\int_{(1/q)^i}^{(1/q)^{i+1}} \lambda^\gamma(t)\, d\mu_{q,\nu}(t)\Big)^{1/\gamma} \leqslant C q^{-i(2\nu+2)(1/\gamma-1)} \int_{(1/q)^{i-1}}^{(1/q)^i} \lambda(t)\, d\mu_{q,\nu}(t).$$

holds. In Lemma 4 below, it is provided that $\lambda(t) = t^\alpha$, $\alpha \in \mathbb{R}$, satisfies this condition.

3. Main results.

Theorem 3. *Let $1 < p \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\nu \geqslant 0$, $f \in L^p_{q,\nu}$, $m > 0$. If $\lambda \in A_{p/(p-pr+r),q,\nu} = A_{p'/(p'-r),q,\nu}$ for some $r \in (0, p')$, $\lambda \in L^{p'/(p'-r)}[0, 1]$ and the integral*

$$\int_1^\infty \lambda(t) t^{-r(2\nu+2)/p'} \omega^r_m(f, t^{-1})_{p,q,\nu} d\mu_{q,\nu}(t)$$

converges, then $\lambda(t)|\mathcal{F}_{q,\nu}(f)(t)|^r \in L^1_{q,\nu}$.

Proof. Let $M_i = [(1/q)^i, (1/q)^{i+1}]$, $i \in \mathbb{Z}_+$. By Lemma 3, (10) and Hausolorf-Young type inequality (8), we have

$$C_1 \int_{M_i} |\mathcal{F}_{q,\nu}(f)(y)|^{p'}\, d\mu_{q,\nu}(y) \leqslant$$

$$\leqslant \int\limits_{M_i} |\mathcal{F}_{q,\nu}(f)(y)|^{p'} (1 - j_\nu(q^i y, q^2))^{mp'}\, d\mu_{q,\nu}(y) \leqslant$$

$$\leqslant \int\limits_0^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'} (1 - j_\nu(q^i y, q^2))^{mp'} d\mu_{q,\nu}(y) = \|\mathcal{F}_{q,\nu}(\Delta^m_{q,\nu,q^i} f)\|^{p'}_{p,q,\nu} \leqslant$$

$$\leqslant C_2 \|\Delta^m_{q,\nu,q^i}(f)\|^{p'}_{p,q,\nu} \leqslant C_2 \omega^{p'}_m(f, q^i)_{p,q,\nu}.$$

By the Hölder inequality and the condition $\lambda \in A_{p'/(p'-r),q,\nu}$, we have for $0 < r < p'$:

$$\int\limits_{M_i} \lambda(t) |\mathcal{F}_{q,\nu}(f)(t)|^r d\mu_{q,\nu}(t) \leqslant$$

$$\leqslant \Big(\int\limits_{M_i} |\lambda(t)|^{p'/(p'-r)}\, d\mu_{q,\nu}(t) \Big)^{1-r/p'} \times \Big(\int\limits_{M_i} |\mathcal{F}_{q,\nu}(f)(t)|^{p'} d\mu_{q,\nu}(t) \Big)^{r/p'} \leqslant$$

$$\leqslant C_3 \omega^r_m(f, q^i)_{p,q,\nu}\; q^{-ir(2\nu+2)/p'} \int\limits_{\mu_{i-1}} \lambda(t)\, d\mu_{q,\nu}(t). \quad (11)$$

By definition,

$$\int\limits_{q^{-i}}^{q^{-i-1}} g(t)\, d\mu_{q,\nu}(t) = (1-q) \Big(q^{n-i-1} \sum_{n=0}^\infty g(q^{n-i-1}) - q^{n-i} \sum_{n=0}^\infty g(q^{n-i}) \Big) =$$

$$= (1-q) \Big(\sum_{n=-i-1}^\infty - \sum_{n=-i}^\infty \Big) q^n g(q^n) = (1-q) q^{-i-1} g(q^{-i-1}). \quad (12)$$

Thus, (11) may be rewritten as

$$\int\limits_{M_i} \lambda(t) |\mathcal{F}_{q,\nu}(f)(t)|^r\, d\mu_{q,\nu}(t) \leqslant C_4 \int\limits_{M_{i-1}} \frac{\lambda(t) \omega_m(f, 1/t)_{p,q,\nu}}{t^{r(2\nu+2)/p'}}\, d\mu_{q,\nu}(t). \quad (13)$$

Summing up (13) over $i = 0, 1, 2, \dots$, we obtain

$$\int\limits_1^\infty \lambda(t) |\mathcal{F}_{q,\nu}(f)(t)|^r d\mu_{q,\nu}(t) \leqslant C_4 \int\limits_{1/q}^\infty \frac{\lambda(t) \omega_m(f, 1/t)_{p,q,\nu}}{t^{r(2\nu+2)/p'}}\, d\mu_{q,\nu}(t). \quad (14)$$

If $\lambda \in L^{p'/(p'-r)}_{q,\nu}[0,1]$, then $\lambda(t) \in L^1[1/q,1]$, and the integral in the right-hand side of (14) is finite, since t^{-1} and $\omega_m(t,1/t)$ are bounded on $[1/q,1]$. Finally, by the condition $\lambda \in L^{p'/(p'-r)}_{q,\nu}[0,1]$ and (8):

$$\int\limits_0^1 \lambda(t)|\mathcal{F}_{q,\nu}(f)(t)|^r\,d\mu_{q,\nu}(t) \leqslant \Big(\int\limits_0^\infty |\mathcal{F}_{q,\nu}(f)(t)|^{p'}\,d\mu_{q,\nu}(t)\Big)^{r/p'} \times \\ \times \Big(\int\limits_0^1 |\mathcal{F}(t)|^{p'/(p'-r)}\,d\mu_{q,\nu}(t)\Big)^{1-r/p'} < \infty.$$

The proof is completed. □

We give two auxiliary statements.

Lemma 4. *A function $\lambda_\alpha(t) = t^\alpha$, $\alpha \in \mathbb{R}$, belongs to any class A_γ, $\gamma \geqslant 1$.*

Proof. By (12), we have

$$I_1 = \Big(\int\limits_{M_i} t^{\alpha\gamma}\,d\mu_{q,\nu}(t)\Big)^{1/\gamma} = ((1-q)q^{-i-1}(q^{-i-1})^{\alpha\gamma+2\nu+1})^{1/\gamma} = \\ = (1-q)^{1/\gamma}q^{-(i+1)(\alpha+(2\nu+2)/\gamma},$$

while

$$I_2 = \int\limits_{M_{i-1}} t^\alpha\,d\mu_{q,\nu}(t) = (1-q)q^{-i(1+\alpha+2\nu+1)},$$

and we obtain $I_1 \leqslant C(q,\nu,\gamma)q^{-i(2\nu+2)(1/\gamma-1)}I_2$. □

Lemma 5. *A function $\lambda_\alpha(t) = t^\alpha$, $a \in \mathbb{R}$, belongs to $L^{p'/(p'-r)}_{q,\nu}[0,1]$, $0 < r < p'$, if and only if $\alpha > -(1-r/p')(2\nu+2)$.*

Proof. By definition, we obtain that the integral

$$\int\limits_0^1 t^{\alpha p'/(p'-r)}d\mu_{q,\nu}(t) = \int\limits_0^1 t^{\alpha p'/(p'-r)+2\nu+1}\,d_{q,\nu}(t) = \\ = (1-q)\sum_{n=0}^\infty q^{n(\alpha p'/(p'-r)+2\nu+2)}$$

converges if and only if $\alpha p'/(p'-r)+2\nu+2>0$ or $\alpha>-(2\nu+2)(1-r/p')$. □

Now we obtain some consequences of the main result.

Corollary 1. *Let $1<p\leqslant 2$, $m>0$, $1/p+1/p'=1$, $\nu\geqslant 0$, $\alpha\in\mathbb{R}$, $f\in L^p_{q,\nu}$, and $r\in(0,p')$. If $\alpha>(r/p'-1)(2\nu+2)$ and the integral*

$$\int_1^\infty t^{\alpha-r(2\nu+2)/p'}\omega_m^r(f,t^{-1})_{p,q,\nu}d\mu_{q\nu}(t) \tag{15}$$

converges, then $t^\alpha|\mathcal{F}_{q,\nu}(f)(t)|^r\in L^1_{q,\nu}$.

Corollary 2. *Let p, p', m, ν and r be as in Corollary 1, and $\omega_m(f,\delta)_{p,q,\nu}=O(\delta^\beta)$, $\beta>0$. If $\alpha>(r/p'-1)(2\nu+2)$ and*

$$p'>r>\frac{p'(\alpha+2\nu+2)}{2\nu+2+p'\beta}, \tag{16}$$

then $t^\alpha|\mathcal{F}_{q,\nu}(f)(t)|^r\in L^1_{q,\nu}$.

Proof. Under conditions of Corollary 2, the integral (15) converges if $\alpha-r(2\nu+2)/p'-r\beta+2\nu+1<-1$ and this inequality is equivalent to (16). □

If $\alpha=0$, then the case $r=q'$ is also admissible. Corollary 3 is an analogue of the Titchmarsh result.

Corollary 3. *Let p, p', m, ν, and r be as in Corollary 1 and $\omega_m(f,\delta)_{p,q,\nu}=O(\delta^\beta)$, $\beta>0$. If*

$$p'\geqslant r>\frac{(2\nu+2)p'}{2\nu+2+p'\beta}=\frac{(2\nu+2)p}{p(2\nu+2+\beta)-(2\nu+2)},$$

then $\mathcal{F}_{q,\nu}(f)\in L^r_{q,\nu}$.

Now we state some estimates for the q-Bessel Fourier transform from q-Besov space.

Theorem 4. *Let $\alpha>0$, $1<p\leqslant 2$, $m>0$, and $\theta\geqslant 1$. If $f\in B^{\alpha,\theta,m}_{p,q,\nu}$, then*

$$\sum_{k\in\mathbb{Z}}q^{-k\theta\alpha}\Big(\sum_{i=k}^\infty\int_{M_i}^\infty|\mathcal{F}_{q,\nu}(f)(y)|^{p'}\Big)^{\theta/p'}<\infty. \tag{17}$$

Proof. In the proof of Theorem 3, it is established that for $i \in \mathbb{Z}$

$$\int_0^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'}(1 - j_\nu(q^i y, q^2))^{mp'}\, d\mu_{q,\nu}(y) \leqslant C_1 \omega_m^{p'}(f, q^i)_{p,q,\nu}.$$

Therefore,

$$\|f\|_B^\theta \geqslant \sum_{k\in\mathbb{Z}} (q^{-k\alpha}\omega_m(f, q^k)_{p,q,\nu})^\theta \geqslant$$

$$\geqslant C_1^{-\theta} \sum_{k\in\mathbb{Z}} q^{-k\theta\alpha} \Big(\int_0^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'}(1 - j_\nu(q^k y, q^2))^{mp'}\, d\mu_{q,\nu}(y) \Big)^{\theta/p'} \geqslant$$

$$\geqslant C_1^{-\theta} \sum_{k\in\mathbb{Z}} q^{-k\theta\alpha} \Big(\int_{q^{-k}}^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'}(1 - j_\nu(q^k y, q^2))^{mp'}\, d\mu_{q,\nu}(y) \Big)^{\theta/p'}.$$

But by Lemma 3 for $y \geqslant q^{-k}$ the inequality $1 - j_\nu(q^k y, q^2) \geqslant C_2 > 0$ holds and one has

$$\|f\|_B^\theta \geqslant C_3 \sum_{k\in\mathbb{Z}} q^{-k\theta\alpha} \Big(\int_{q^{-k}}^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'}\, d\mu_{q,\nu}(y) \Big)^{\theta/p'}, \tag{18}$$

that is equivalent to (17). □

Corollary 4 is an analogue of the famous theorem of Herz for the classical Fourier transform (see [10]). The proof in our case is simpler.

Corollary 4. *Let $\alpha > 0$, $1 < p \leqslant 2$, $m > 0$, and $\theta \geqslant 1$. If $f \in B^{\alpha,\theta,m}_{p,q,\nu}$, then $\mathcal{F}_{q,\nu}(f) \in K(\alpha, \theta, p, q, \nu)$ and $\|\mathcal{F}_{q,\nu}(f)\|_{K(\alpha,\theta,p,q,\nu)} \leqslant C\|f\|_B$.*

Proof. The result follows from (18) and the obvious inequality

$$\int_{q^{-k}}^\infty |\mathcal{F}_{q,\nu}(f)(y)|^{p'}\, d\mu_{q,\nu}(y) \geqslant \int_{q^{-k}}^{q^{-k-1}} |\mathcal{F}_{q,\nu}(f)(y)|^{p'}\, d\mu_{q,\nu}(y).$$

The proof is completed. □

References

[1] Achak A., Daher R., Dhaouadi L., Loualid E.M. *An analog of Titchmarsh's theorem for the q-Bessel transform.* Ann. Univ. Ferrara., 2019, vol. 65, no. 1, pp. 1–13. DOI: https://doi.org/10.1007/s11565-018-0309-3

[2] Bergh J., Löfström J. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin-Heidelberg, 1976.

[3] Butzer P. L., Nessel R. J. *Fourier analysis and approximation.* Birkhauser, Basel-Stuttgart, 1971.

[4] Butzer P. L., Dyckhoff H., Gorlich E., Stens R. L. *Best trigonometric approximation, fractional order derivatives and Lipschitz classes.* Can. J. Math., 1977, vol. 29, no. 4, pp. 781–793.

[5] Daher R., Tyr O. *Growth properties of the q-Dunkl transform in the space* $L^p_{q,\alpha}(\mathbb{R}_q, |x|^{2\alpha+1}\, d_q(x))$. Ramanujan J. (accepted). DOI: https://doi.org/10.1007/s11139-021-00387-x

[6] Dhaouadi L., Fitouhi A., El Kamel J. *Inequalities in q-Fourier analysis.* J. Ineq. Pure Appl. Math., 2006, vol. 7, no. 5. Art. 171.

[7] Dhaouadi L. *On the q-Bessel Fourier transform.* Bull. Math. Anal. Appl., 2013, vol. 5, no. 2, pp. 42 – 60.

[8] Fitouhi A., Dhaouadi L. *Positivity of the generalized translation associated with the q-Hankel transform.* Const. Approx., 2011, vol. 34, no. 3, pp. 435 – 472. DOI: https://doi.org/10.1007/s00365-011-9132-0

[9] Gogoladze L., Meskhia R. *On the absolute convergence of trigonometric Fourier series.* Proc. Razmadze Math. Inst., 2006, vol. 141, pp. 29 – 46.

[10] Herz C. *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms.* J. Math. Mech., 1968, vol. 18, no. 2, pp. 283 – 324.

[11] Kac V., Cheung P. Quantum calculus. Springer, New York, 2002.

[12] Koornwinder T.H., Swarttouw F.H. *On q-analogues of the Hankel and Fourier transform.* Trans. Amer. Math. Soc., 1992, vol. 333, no. 1, pp. 445 – 461.

[13] Móricz F. *Sufficient conditions for the Lebesgue integrability of Fourier transforms.* Anal. Math., 2010, vol. 36, no. 2, pp. 121 – 129.

[14] Platonov S. S. *On the Hankel transform of functions from Nikol'skii classes.* Integral Transforms Spec. Funct., 2021, vol. 32, no. 10, pp. 823 – 838. DOI: https://doi.org/10.1080/10652469.2020.1849184

[15] Szasz O. *Fourier series and mean moduli of continuity.* Trans. Amer. Math. Soc., 1937, vol. 42, no. 3, pp. 366 – 395.

[16] Titchmarsh E. *Introduction to the theory of Fourier integrals.* Clarendon press, Oxford, 1937.

[17] Volosivets S. *Weighted integrability of Fourier-Dunkl transforms and generalized Lipschitz classes.* Analysis Math. Phys., 2022, vol. 12, paper 115. DOI: https://doi.org/10.1007/s13324-022-00728-z

[18] Volosivets S. S. *Weighted integrability results for first Hankel- Clifford transform*, Probl. Anal. Issues Anal., 2023, vol. 12(30), no. 2, 2023, pp. 107 – 117. DOI: https://doi.org/10.15393/j3.art.2023.13050

[19] Younis M. S. *Fourier transforms of Dini-Lipschitz functions.* Int. J. Math. Math. Sci., 1986, vol. 9(2), pp. 301 – 312.

Received August 19, 2023.
In revised form, November 30, 2023.
Accepted February 15, 2024.
Published online February 23, 2024.

Saratov State University
83 Astrakhanskaya St., Saratov 410012, Russia
E-mail: julia.krotova.sgu@gmail.com