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A NEW APPROACH TO JAGGI–WARDOWSKI-TYPE FIXED POINT THEOREMS

Abstract. In this manuscript, the concept of Jaggi-type hybrid (G, φ, F)-contraction is introduced. Some novel fixed point theorems that cannot be inferred from their cognate ones in both metric and quasi metric spaces are established. A non-trivial example is also provided to support the assumptions forming our obtained theorems. As an application, one of our results is utilized to study new conditions for the existence and uniqueness of a solution to a Fredholm-type integral equation.

Key words: *F-contraction, G-metric, fixed point, hybrid contraction*

2020 Mathematical Subject Classification: *47H10, 55M20, 54H25*

1. Introduction. In 1922, Banach [5] proposed the well-known fixed point result called Banach contraction principle, which provided the groundwork for metric fixed point theory. Due to the simplicity and usefulness of the contraction mapping principle, it has been extended in different directions. One of such generalizations, involving some rational expressions, was presented by Jaggi [12]. For similar improvements, one can also refer to [4], [8], [7], [27], and some references therein.

The mathematical community has found research on new spaces and their properties to be an intriguing topic over time. One of such developments is the idea of G-metric space (or generalized metric space) introduced by Mustafa and Sims [20]. The aim of G-metric space was to correct some flaws in the idea of generalized metric space given by Dhage [10], and to adequately depict the geometry of three points along the perimeter of a triangle. Consequently, Mustafa et al. [21] proved some fixed point results of Hardy-Rogers-type for mappings satisfying certain constraints on a complete metric space. However, according to the works of Jleli et al. [14] as well as Samet et al. [24], several fixed point results in the framework of G-metric space can be deduced from some existing results in quasi-metric space. It was noted in [6], [16] that the former observation is appropriate only if the contractive conditions in the result can be reduced to two variables. We refer to Jiddah et al. [13] for an extensive survey on the advancements of fixed point results in G-metric space.

In 2012, Wardowski [30] enhanced the Banach contraction principle by introducing a new concept of contractions known as F-contraction and proved a fixed point theorem. Then, Wardowski and Dung [31] developed the idea of an F-weak contraction on a metric space and attained a generalization of F-contraction. Secelean [25] pointed out that the condition $(F2)$ which states that: for every sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$, in Wardowski's formulation of F-contraction can be replaced with a more straightforward and equivalent condition provided by $(F2')$: $\inf F = -\infty$. Piri and Kumam [23] in 2014 extended the result of Wardowski [30] by enforcing weaker auxiliary conditions on a self map defined on a complete metric space. Cosentino and Vetro [9] established some fixed point results of Hardy-Rogers-type for self mappings on complete metric spaces and ordered metric spaces. Thereafter, Wardowski [32] suggested the replacement of the positive constant τ in the original formulation of F-contraction with a function satisfying certain constraints, and, hence, established a new type of contractions on metric spaces called (φ, F)-contractions. Not long ago, Lotfali et al. [17] established some new fixed point theorems involving set-valued F-contractions in the setting of quasi-ordered metric spaces. Amine et al. [3] presented a new concept of μF-contraction in Banach spaces, by modeling (with measure of non-compactness) the idea of F-contraction initiated by Wardowski [30]. For more recent trends in F-contraction type fixed point results, see Joshi and Jain [15], Fabiano et al. [11], Secelean [26], Touail et al. [29] and some references therein.

Agarwal et al. [1] studied the existence and uniqueness of invariant points of a family of $(\varphi,\ F)$-contraction in the context of metric-like spaces and used their findings to ascertain some conditions for the existence of solutions of Fredholm integral equations on time scales. Mohammed et al. [18] enhanced the work of [1] by investigating a class of $(\varphi,\ F)$-weak contractions in metric-like spaces under the same conditions. Inspired by these developments, we introduce a new concept called Jaggi-type hybrid $(G,\ \varphi,\ F)$-contraction in the framework of G-metric space and study new conditions under which such contractions have at least a fixed point. A non-trivial example is provided to support the applicability of our results and its refinement over previous findings.

The paper is structured as follows: The introduction and synopsis of relevant literature are presented in Section 1. Section 2 compiles the basic ideas required in this work. In Section 3, the main results and a few corollaries acquired from our findings are presented.

2. Preliminaries. In this section, we present some basic notations and results to be used subsequently.

Throughout this paper, the set X is considered non-empty. We denote by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ the set of real numbers, the set of non-negative real numbers, and the set of natural numbers, respectively.

Definition 1. [20] *Let X be a non-empty set and $G\colon X \times X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying:*

- (G_1) $G(x,y,z) = 0$ *if* $x = y = z$;
- (G_2) $0 < G(x,x,y)$ *for all* $x, y \in X$ *with* $x \neq y$;
- (G_3) $G(x,x,y) \leqslant G(x,y,z)$, *for all* $x,y,z \in X$ *with* $z \neq y$;
- (G_4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = \ldots$ *(symmetry in all three variables)*;
- (G_5) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$, *for all* $x,y,z,a \in X$ *(rectangle inequality)*.

Then the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Example. [21] Let (X,d) be a usual metric space. Then (X,G_s) and (X,G_m) are G-metric spaces, where

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z) \quad \forall x,y,z \in X, \tag{1}$$
$$G_m(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\} \quad \forall x,y,z \in X. \tag{2}$$

Definition 2. [21] *Let (X,G) be a G-metric space and $\{x_n\}$ be a sequence of points of X. We say that sequence $\{x_n\}$ is G-convergent to x if $\lim\limits_{n,m\to\infty} G(x,x_n,x_m) = 0$; that is, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \geqslant n_0$. We refer to x as the limit of the sequence $\{x_n\}$.*

Proposition 1. *[21] Let (X, G) be a G-metric space. Then the following are equivalent:*

(i) $\{x_n\}$ *is G-convergent to x.*

(ii) $G(x, x_n, x_m) \longrightarrow 0$, *as* $n, m \to \infty$.

(iii) $G(x_n, x, x) \longrightarrow 0$, *as* $n \to \infty$.

(iv) $G(x_n, x_n, x) \longrightarrow 0$, *as* $n \to \infty$.

Definition 3. [21] *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geqslant n_0$; that is, $G(x_n, x_m, x_l) \longrightarrow 0$, as $n, m, l \to \infty$.*

Proposition. *[21] In a G-metric space (X, G), the following are equivalent:*

(i) *The sequence $\{x_n\}$ is G-Cauchy.*

(ii) *For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geqslant n_0$.*

Definition 4. [21] *Let (X, G) and (X', G') be G-metric spaces and $f\colon (X, G) \longrightarrow (X', G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$, such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.*

Proposition. *[21] Let (X, G) and (X', G') be two G-metric spaces. Then a function $f\colon (X, G) \longrightarrow (X', G')$ is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to fx.*

Definition 5. [21] *A G-metric space (X, G) is called symmetric G-metric space if $G(x, x, y) = G(y, x, x) \quad \forall\, x, y \in X$.*

Proposition. *[21] Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition. *[21] Every G-metric space (X, G) will define a metric space (X, d_G) by $d_G(x, y) = G(x, y, y) + G(y, x, x)\ \forall x, y \in X$. Note that if (X, G) is a symmetric G-metric space, then $(X, d_G) = 2G(x, y, y)\ \forall x, y \in X$. However, if (X, G) is not symmetric, then it holds due to G-metric properties that $\frac{3}{2}G(x, y, y) \leqslant d_G(x, y) \leqslant 3G(x, y, y) \quad \forall\, x, y \in X$.*

Definition 6. [21] *A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Proposition. *[21] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Mustafa [19] proved the following result in the framework of G-metric space.

Theorem 1. *[19] Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a mapping satisfying the following condition:*

$$G(Tx, Ty, Tz) \leqslant kG(x, y, z), \tag{3}$$

for all $x, y, z \in X$, where $0 \leqslant k < 1$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G-continuous at u.

Following in the direction of [30], the idea of F-contraction is defined as follows:

Definition 7. [30] *Let Δ_f denote the family of functions $F\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following auxiliary conditions:*

- $(F1)$ *F is strictly increasing; that is, for all a, $b \in \mathbb{R}_+$, if $a < b$ then $F(a) < F(b)$;*
- $(F2)$ *for every sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$;*
- $(F3)$ *there exists $0 < k < 1$, such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Definition 8. [30] *Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction if there exist $\tau > 0$ and $F \in \Delta_f$, such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y)). \tag{4}$$

Remark. *From (F1) and (2.7), it is clear that if T is an F-contraction, then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$, such that $Tx \neq Ty$. That is, T is a contractive mapping and, hence, every F-contraction is a continuous mapping.*

Wardowski [30] presented a variation of the Banach fixed point theorem as follows:

Theorem 2. *Let (X, d) be a complete metric space and $T\colon X \longrightarrow X$ be an F-contraction. Then T has a unique fixed point $x \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x.*

A modification of the class of contractions called F-contraction was proposed by Wardowski [32], where the positive constant τ was replaced by a function φ and some of the conditions of F were relaxed. The modified version of F-contraction is presented as follows:

Definition 9. [32] *Let (X, d) be a metric space, $F\colon (0, \infty) \to \mathbb{R}$ and $\varphi\colon (0, \infty) \to (0, \infty)$ be two mappings satisfying the following:*

(1) *F is strictly increasing, i.e., $\alpha < \beta$ implies $F(\alpha) < F(\beta)$ for all $\alpha, \beta \in (0, \infty)$;*

(2) $\lim\limits_{\alpha\to 0^+} F(\alpha) = -\infty$;

(3) $\liminf\limits_{\alpha\to t^+} \varphi(\alpha) > 0$ *for all $t > 0$.*

A mapping $T\colon X \to X$ is called a (φ, F)-contraction on (X, d) if

$$\varphi(d(x, y)) + F(d(Tx, Ty)) \leqslant F(d(x, y)),$$

for all $x, y \in X$ for which $Tx \neq Ty$.

In line with [28], let Φ be the set of all functions $\phi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, such that ϕ is a non-decreasing function with $\lim\limits_{n\to+\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then ϕ is called a Φ-map.

Let $\phi \in \Phi$ be a Φ-map, such that there exist $n_0 \in \mathbb{N}$, $k \in (0, 1)$, and a convergent series of non-negative terms $\sum_{n=1}^{\infty} v_n$ satisfying $\phi^{n+1}(t) \leqslant k\phi^n(t) + v_n$ for $n \geqslant n_0$ and any $t > 0$. Then ϕ is called a (c)-comparison function [2].

Lemma 1. *[2] If $\phi \in \Phi$, then the following hold:*

(i) *$\{\phi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for $t \geqslant 0$;*

(ii) *$\phi(t) < t$ for any $t \in \mathbb{R}_+$;*

(iii) *ϕ is continuous;*

(iv) *$\phi(t) = 0$ if and only if $t = 0$;*

(v) *the series $\sum_{i=1}^{\infty} \phi^i(t)$ is convergent for $t \geqslant 0$.*

3. Main results. In this section, the concept of (φ, F)-type contractions is initiated in line with Definition 9 on G-metric space. Conditions for the existence and uniqueness of invariant points for such family of contractions are studied.

Definition 10. *Let (X, G) be a G-metric space. A mapping $T\colon X \longrightarrow X$ is called a Jaggi-type hybrid (G, φ, F)-contraction of type (A) if there exist $F \in \Delta_f$, $\phi \in \Phi$ and a function $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying condition 3 of definition 9, such that $G(Tx, Ty, T^2y) > 0$ implies*

$$\varphi(G(x, y, Ty)) + F(G(Tx, Ty, T^2y)) \leqslant F(\phi(\mathcal{R}_A^q(x, y, Ty))), \tag{5}$$

for all $x, y \in X \backslash Fix(T)$, where

$$\mathcal{R}_A^q(x, y, Ty) = \begin{cases} \left[\lambda_1\Big(\dfrac{G(x, Tx, T^2x) \cdot G(y, Ty, T^2y)}{G(x, y, Ty)}\Big)^q + \lambda_2 G(x, y, Ty)^q\right]^{\frac{1}{q}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad for\ some\ q > 0; \\ \\ G(x, Tx, T^2x)^{\lambda_1} \cdot G(y, Ty, T^2y)^{\lambda_2}, \quad for \quad q = 0, \end{cases} \tag{6}$$

$\lambda_1, \lambda_2 \geqslant 0$ with $\lambda_1 + \lambda_2 = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

We present our main results as follows:

Theorem 3. *Let (X, G) be a complete G-metric space. If the mapping $T\colon X \longrightarrow X$ is a continuous Jaggi-type hybrid (G, φ, F)-contraction of type (A) on (X, G), then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary, but fixed. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_n = T^n x_0$. If we can find some $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is complete. Suppose that $Tx_{n-1} \neq Tx_n$ for all $n \in \mathbb{N}$ and $G(Tx_{n-1}, Tx_n, T^2x_n) > 0$. Since T is a Jaggi-type hybrid (G, φ, F)-contraction, from (5) and taking $x = x_{n-1}$, $y = x_n$, we have

$$\begin{aligned} \varphi(G(x_{n-1}, x_n, Tx_n)) + F(G(Tx_{n-1}, Tx_n, T^2x_n)) \leqslant \\ \leqslant F(\phi(\mathcal{R}_A^q(x_{n-1}, x_n, Tx_n))). \end{aligned} \tag{7}$$

We then consider the following cases of (6):

Case 1: For $q > 0$, we obtain:

$$\mathcal{R}_A^q(x_{n-1}, x_n, Tx_n) = \Big[\lambda_1\Big(\frac{G(x_{n-1}, Tx_{n-1}, T^2x_{n-1})G(x_n, Tx_n, T^2x_n)}{G(x_{n-1}, x_n, Tx_n)}\Big)^q +$$

$$
\begin{aligned}
&+\lambda_2 G(x_{n-1},x_n,Tx_n)^q\Big]^{\frac{1}{q}} = \\
&= \Big[\lambda_1\Big(\frac{G(x_{n-1},x_n,x_{n+1})G(x_n,x_{n+1},x_{n+2})}{G(x_{n-1},x_n,x_{n+1})}\Big)^q + \\
&+\lambda_2 G(x_{n-1},x_n,x_{n+1})^q\Big]^{\frac{1}{q}} = \\
&= \ \big[\lambda_1 G(x_n,x_{n+1},x_{n+2})^q + \lambda_2 G(x_{n-1},x_{n,n+1})^q\big]^{\frac{1}{q}}.
\end{aligned} \tag{8}
$$

Assume that $G(x_{n-1},x_n,x_{n+1}) \leqslant G(x_n,x_{n+1},x_{n+2})$ for all $n \in \mathbb{N}$. Since ϕ is nondecreasing, then from (8) the contractive inequality (7) takes the form

$$
\begin{aligned}
&F(G(x_n,x_{n+1},x_{n+2})) \leqslant F(\phi(\mathcal{R}_A^q(x_{n-1},x_n,Tx_n))) - \varphi(G(x_{n-1},x_n,Tx_n)) = \\
&\quad = F(\phi\,[\lambda_1 G(x_n,x_{n+1},x_{n+2})^q + \lambda_2 G(x_{n-1},x_n,x_{n+1})^q]^{\frac{1}{q}}) - \\
&\qquad - \varphi(G(x_{n-1},x_n,x_{n+1})) \leqslant \\
&\quad \leqslant F(\phi\,[\lambda_1 G(x_n,x_{n+1},x_{n+2})^q + \lambda_2 G(x_n,x_{n+1},x_{n+2})^q]^{\frac{1}{q}}) - \\
&\qquad - \varphi(G(x_n,x_{n+1},x_{n+2})) = \\
&\quad = F(\phi[(\lambda_1+\lambda_2)^{\frac{1}{q}} G(x_n,x_{n+1},x_{n+2})]) - \varphi(G(x_n,x_{n+1},x_{n+2})) = \\
&\quad = F(\phi(G(x_n,x_{n+1},x_{n+2}))) - \varphi(G(x_n,x_{n+1},x_{n+2})) < \\
&\quad < F(G(x_n,x_{n+1},x_{n+2})), \quad \text{which is not true.}
\end{aligned}
$$

Therefore, $\max\{G(x_{n-1},x_n,x_{n+1},G(x_n,x_{n+1},x_{n+2}))\} = G(x_{n-1},x_n,x_{n+1})$, so that (7) becomes

$$
\begin{aligned}
&F(G(x_n,x_{n+1},x_{n+2})) \leqslant \\
&\qquad \leqslant F(\phi\,[\lambda_1 G(x_n,x_{n+1},x_{n+2})^q + \lambda_2 G(x_{n-1},x_n,x_{n+1})^q]^{\frac{1}{q}}) - \\
&\qquad\qquad - \varphi(G(x_{n-1},x_n,x_{n+1})) \leqslant \\
&\qquad \leqslant F(\phi[(\lambda_1+\lambda_2)^{\frac{1}{q}} G(x_{n-1},x_n,x_{n+1}]) - \varphi(G(x_{n-1},x_n,x_{n+1})) \leqslant \\
&\qquad\qquad \leqslant F(\phi(G(x_{n-1},x_n,x_{n+1}))) - \varphi(G(x_{n-1},x_n,x_{n+1})).
\end{aligned} \tag{9}
$$

Let $\gamma_n = G(x_n,x_{n+1},x_{n+2}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From (9) and since F is increasing, $\gamma_n \leqslant \gamma_{n-1}$ for all $n \in \mathbb{N}$. That is, the positive sequence $\{\gamma_n\}$ is decreasing and thus converges to a limit $\gamma \geqslant 0$. The inequality (9) yields

$$
F(\gamma_n) \leqslant F(\phi(\gamma_{n-1})) - \varphi(\gamma_{n-1}) \leqslant
$$

$$\begin{aligned}
&\leqslant F(\phi^2(\gamma_{n-2})) - \varphi(\gamma_{n-2}) - \varphi(\gamma_{n-1}) \leqslant \\
&\vdots \\
&\leqslant F(\phi^n(\gamma_0)) - \sum_{i=0}^{n-1} \varphi(\gamma_i).
\end{aligned} \tag{10}$$

Owing to the fact that from Condition 3 of Definition 9, $\liminf_{\alpha \to t^+} \varphi(\alpha) > 0$ for all $t > 0$, we have $\liminf_{n\to\infty} \varphi(\gamma_n) > 0$. Therefore, we can find $n_0 \in \mathbb{N}$ and $a > 0$, such that for all $n \geqslant n_0$, $\varphi(\gamma_n) > a$. Thus, the inequality (10) becomes

$$\begin{aligned}
F(\gamma_n) \leqslant F(\phi^n(\gamma_0)) - \sum_{i=0}^{n_0-1} \varphi(\gamma_i) - \sum_{i=n_0}^{n-1} \varphi(\gamma_i) \leqslant F(\phi^n(\gamma_0)) - \sum_{i=n_0}^{n-1} a \leqslant \\
\leqslant F(\phi^n(\gamma_0)) - (n - n_0)a \quad \text{for all } n \geqslant n_0.
\end{aligned} \tag{11}$$

Applying limit as $n \to \infty$ in (11), we obtain

$$\lim_{n\to\infty} F(\gamma_n) \leqslant \lim_{n\to\infty} [F(\phi^n(\gamma_0)) - (n - n_0)a] = -\infty.$$

Since $\lim_{n\to\infty} F(\gamma_n) = -\infty$, then from (F2) it follows that $\lim_{n\to\infty} \gamma_n = 0$. From $(F3)$, we can find $k \in (0,1)$, such that $\lim_{n\to\infty} {\gamma_n}^k F(\gamma_n) = 0$. By (11), the following is valid for all $n \in \mathbb{N}$: ${\gamma_n}^k F(\gamma_n) \leqslant {\gamma_n}^k [F(\phi^n(\gamma_0)) - (n - n_0)a]$, and it follows that ${\gamma_n}^k [F(\gamma_n) - F(\phi^n(\gamma_0))] \leqslant -(\gamma_n)^k (n - n_0)a \leqslant 0$. Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} (\gamma_n)^k (n - n_0)a = 0. \tag{12}$$

From (12), we can find $n_1 \in \mathbb{N}$, such that $(\gamma_n)^k (n - n_0)a \leqslant 1$ for all $n \geqslant n_1$. Thus, we obtain

$$\gamma_n \leqslant \frac{1}{[(n - n_0)a]^{\frac{1}{k}}}, \quad \forall\, n \geqslant n_1.$$

In order to show that $\{x_n\}$ is a G-Cauchy sequence, consider $m, n \in \mathbb{N}$, such that $m > n \geqslant \max\{n_0, n_1\}$. Therefore, we have

$$\begin{aligned}
&G(x_n, x_n, x_m) \leqslant \\
&\quad \leqslant G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{m-1}, x_m, x_m) =
\end{aligned}$$

$$= \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1} = \sum_{i=n}^{m-1} \gamma_i \leqslant \sum_{i=n}^{\infty} \gamma_i \leqslant \sum_{i=n}^{\infty} \frac{1}{[(n-n_0)a]^{\frac{1}{k}}} \ . \quad (13)$$

Since $0 < k < 1$, the infinite series $\sum_{i=n}^{\infty} \frac{1}{[(i-n_0)a]^{\frac{1}{k}}}$ is a G-Cauchy sequence in (X, G) and, so, by the completeness of (X, G), we can find such $x^* \in X$, that $\{x_n\}$ converges to x^*. That is, $\lim_{n\to\infty} G(x_n, x^*, x^*) = 0$. By the assumption that T is continuous, $\lim_{n\to\infty} G(x^*, x^*, Tx^*) = \lim_{n\to\infty} G(x_{n+1}, x_{n+1}, Tx^*) = 0$, which implies that $Tx^* = x^*$, that is, x^* is a fixed point of T.

Case 2: For $q = 0$, we have

$$\begin{aligned} \mathcal{R}_A^q(x_{n-1}, x_n, Tx_n) &= G(x_{n-1}, Tx_{n-1}, T^2x_{n-1})^{\lambda_1} G(x_n, Tx_n, T^2x_n)^{\lambda_2} = \\ &= G(x_{n-1}, x_n, x_{n+1})^{\lambda_1} G(x_n, x_{n+1}, x_{n+2})^{\lambda_2}. \end{aligned} \quad (14)$$

If $G(x_{n-1}, x_n, x_{n+1}) \leqslant G(x_n, x_{n+1}, x_{n+2})$, then, using (14) in (7), we have

$$\begin{aligned} F(G(Tx_{n-1},\ &Tx_n, T^2x_n)) \leqslant \\ &\leqslant F(\phi(\mathcal{R}_A^q(x_{n-1}, x_n, Tx_n))) - \varphi(G(x_{n-1}, x_n, Tx_n)) = \\ &= F(\phi[G(x_{n-1}, x_n, x_{n+1})^{\lambda_1} G(x_n, x_{n+1}, x_{n+2})^{\lambda_2}]) \\ &- \varphi(G(x_{n-1}, x_n, x_{n+1})) \leqslant \\ &\leqslant F(\phi[G(x_n, x_{n+1}, x_{n+2})^{\lambda_1} G(x_n, x_{n+1}, x_{n+2})^{\lambda_2}]) - \\ &- \varphi(G(x_n, x_{n+1}, x_{n+2})) = \\ &= F(\phi[G(x_n, x_{n+1}, x_{n+2})^{(\lambda_1+\lambda_2)}]) - \varphi(G(x_n, x_{n+1}, x_{n+2})) = \\ &= F(\phi(G(x_n, x_{n+1}, x_{n+2}))) - \varphi(G(x_n, x_{n+1}, x_{n+2})) < \\ &< F(G(x_n, x_{n+1}, x_{n+2})). \end{aligned}$$

That is, $F(G(x_n, x_{n+1}, x_{n+2})) < F(G(x_n, x_{n+1}, x_{n+2}))$, which is not true. Therefore, $G(x_n, x_{n+1}, x_{n+2}) < G(x_{n-1}, x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Consequently, we have

$$F(G(x_n, x_{n+1}, x_{n+2})) \leqslant F(\phi(G(x_{n-1}, x_n, x_{n+1}))) - \varphi(G(x_{n-1}, x_n, x_{n+1})).$$

By a similar method as in the case of $q > 0$, we can establish that there is a G-Cauchy sequence $\{x_n\}$ in (X, G) and a point $x^* \in X$, such that $\lim_{n\to\infty} x_n = x^*$. Similarly, under the assumption that T is continuous and by uniqueness of limit, we have $Tx^* = x^*$. That is, x^* is a fixed point of T.

To show uniqueness, assume that T has two distinct fixed points x^* and y^* in X, such that $Tx^* = x^* \neq y^* = Ty^*$. Then the contractive inequality (5) yields

$$\begin{aligned}&\varphi(G(x^*, y^*, Ty^*)) + F(G(Tx^*, Ty^*, T^2y^*)) = \\ &\quad = \varphi(G(x^*, y^*, y^*)) + F(G(x^*, y^*, y^*)) \leqslant F(\phi(\mathcal{R}_A^q(x^*, y^*, Ty^*))). \quad (15)\end{aligned}$$

where there are two cases.

Case 1: For $q > 0$

$$\begin{aligned}&\mathcal{R}_A^q\ (x^*, y^*, Ty^*) = \\ &= \Big[\lambda_1\Big(\frac{G(x^*, Tx^*, T^2x^*)G(y^*, Ty^*, T^2x^*)}{G(x^*, y^*, Ty^*)}\Big)^q + \lambda_2(G(x^*, y^*, Ty^*))^q\Big]^{\frac{1}{q}} = \\ &= \Big[\lambda_1\Big(\frac{G(x^*, x^*, x^*)G(y^*, y^*, y^*)}{G(x^*, y^*, y^*)}\Big)^q + \lambda_2(G(x^*, y^*, y^*))^q\Big]^{\frac{1}{q}} = \\ &= \big[\lambda_2(G(x^*, y^*, y^*))^q\big]^{\frac{1}{q}} = \lambda_2^{\frac{1}{q}} G(x^*, y^*, y^*) \leqslant\ G(x^*, y^*, y^*).\end{aligned}$$

Hence, (15) takes the form

$$\begin{aligned}\varphi(G(x^*, y^*, Ty^*)) + F(G(x^*, y^*, y^*)) &\leqslant F(\phi(G(x^*, y^*, y^*))) < \\ &< F(G(x^*, y^*, y^*)),\end{aligned}$$

which is a contradiction, since $\varphi(G(x^*, y^*, Ty^*)) > 0$.

Case 2: For $q = 0$

$$\begin{aligned}\mathcal{R}_A^q(x^*, y^*, Ty^*) &= G(x^*, Tx^*, T^2x^*)^{\lambda_1}.G(y^*, Ty^*, T^2y^*)^{\lambda_2} = \\ &= G(x^*, x^*, x^*)^{\lambda_1}.G(y^*, y^*, y^*)^{\lambda_2} = 0.\end{aligned}$$

Therefore, from (15) we have

$$\varphi(G(x^*, y^*, Ty^*)) + F(G(x^*, y^*, y^*)) \leqslant F(\phi(0)) < F(0).$$

From $(G2)$ and $(F1)$, we observe that $0 < G(x^*, y^*, y^*)$ implies $F(0) < F(G(x^*, y^*, y^*))$. It follows that

$$\varphi(G(x^*, y^*, Ty^*)) < F(0) - F(G(x^*, y^*, y^*)) < 0,$$

which is a contradiction. Thus, we can conclude that $x^* = y^*$. That is, T has only one fixed point in (X, G). This completes the proof. $\square$

Definition 11. *Let (X,G) be a G-metric space. A mapping $T\colon X \longrightarrow X$ is called a Jaggi-type hybrid $(G,\,\varphi,\,F)$-contraction of type (B) if there exist $F \in \Delta_f$, $\phi \in \Phi$ and a function $\varphi\colon (0,\infty) \to (0,\infty)$ satisfying Condition 3 of Definition 9, such that $G(Tx,Ty,T^2y) > 0$ implies*

$$\varphi(G(x,y,Ty)) + F(G(Tx,Ty,T^2y)) \leqslant F(\phi(\mathcal{R}_B^q(x,y,Ty))), \tag{16}$$

for all $x,y \in X$, where

$$\mathcal{R}_B^q(x,y,Ty) = \left[\lambda_1\Big(\frac{G(x,Tx,T^2x)\cdot G(y,Ty,T^2y)}{G(x,y,Ty)}\Big)^q + \lambda_2 G(x,y,Ty)^q\right]^{\frac{1}{q}} \tag{17}$$

$q > 0$; $\lambda_1,\lambda_2 \geqslant 0$ with $\lambda_1+\lambda_2 = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

Definition 12. *Let (X,G) be a G-metric space. A mapping $T\colon X \longrightarrow X$ is called a Jaggi-type hybrid $(G,\,\varphi,\,F)$-contraction of type (C) if there exist $F \in \Delta_f$, $\phi \in \Phi$, and a function $\varphi\colon (0,\infty) \to (0,\infty)$ satisfying Condition 3 of Definition 9, such that $G(Tx,Ty,T^2y) > 0$ implies*

$$\varphi(G(x,y,Ty)) + F(G(Tx,Ty,T^2y)) \leqslant F(\phi(\mathcal{R}_C(x,y,Ty))),$$

for all $x,y \in X\backslash Fix(T)$, where

$$\mathcal{R}_C(x,y,Ty) = \left[G(x,Tx,T^2x)^{\lambda_1}\cdot G(y,Ty,T^2y)^{\lambda_2}\right] \tag{18}$$

$\lambda_1,\lambda_2 \geqslant 0$ with $\lambda_1+\lambda_2 = 1$ and $Fix(T) = \{x \in X : Tx = x\}$.

The following are some immediate cases of our results:

Corollary 1. *Let (X,G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a Jaggi-type hybrid $(G,\,\varphi,\,F)$-contraction of type (B). If T is continuous, then T possesses a unique fixed point in X.*

Proof. Following the proof of Theorem 3 for the case where $q > 0$, the conclusion follows. □

Corollary 2. *Let (X,G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a Jaggi-type hybrid $(G,\,\varphi,\,F)$-contraction of type (C). If T is continuous, then T has a unique fixed point in X.*

Proof. It can be easily deduced from the proof of Theorem 3 by considering the case where $q = 0$. □

Corollary 3. *Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping, such that*

$$\varphi(G(x, y, Ty)) + F(G(Tx, Ty, T^2 y)) \leqslant F(\phi(G(x, y, Ty))),$$

for all $x, y \in X, F \in \Delta_f$ and $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying condition 3 of definition 9. Then T has a unique fixed point in X.

Proof. Considering the inequality (5) for $q > 0$, let $\lambda_1 = 0$ and $\lambda_2 = 1$; the proof follows. □

Corollary. *Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping, such that*

$$\varphi(G(x, y, z)) + F(G(Tx, Ty, Tz)) \leqslant F(\phi(G(x, y, z))),$$

for all $x, y, z \in X, F \in \Delta_f$ and $\varphi\colon (0, \infty) \to (0, \infty)$, satisfying condition 3 of definition 9. Then T possesses a unique fixed point in X.

Proof. Take $Ty = z$ in Corollary 3. □

Corollary 4. *Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping, such that*

$$\varphi(G(x, y, Ty)) + F(G(Tx, Ty, T^2 y)) < F(\mathcal{R}_A^q(x, y, Ty)),$$

$\forall\ x, y \in X,\ F \in \Delta_f$ and $\varphi\colon (0, \infty) \to (0, \infty)$ satisfying condition 3 of definition 9. Then T has a unique fixed point in X.

Proof. Defining $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(t) = \eta t$ for $t > 0$ and $\eta \in (0, 1)$ in Theorem 3, we have

$$\begin{aligned}\varphi(G(x, y, Ty)) + F(G(Tx, Ty, T^2 y)) &\leqslant \\ &\leqslant F(\eta(\mathcal{R}_A^q(x, y, Ty))) < F(\mathcal{R}_A^q(x, y, Ty)).\end{aligned}$$

□

Corollary 5. *Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping. If there exists a constant $\tau > 0$, such that*

$$\tau + F(G(Tx, Ty, T^2 y)) < F(\mathcal{R}_A^q(x, y, Ty)), \tag{19}$$

for all $x, y \in X$, $F \in \Delta_f$ and $\mathcal{R}_A^q(x, y, Ty)$ is as given in Definition 6, then T has a unique fixed point in X.

Proof. Taking $\varphi(t) = \tau > 0$ for all $t > 0$ in Corollary 4, the result follows. □

Corollary 6. *(see [22], Theorem 3) Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping. If there exists a constant $\tau > 0$, such that*

$$\tau + F(G(Tx, Ty, T^2y)) \leqslant F(\phi(\mathcal{R}_A^q(x, y, Ty))),$$

for all $x, y \in X$, $\phi \in \Phi$, $F \in \Delta_f$ and $\mathcal{R}_A^q(x, y, Ty)$ is as given in Definition 6, then T has a unique fixed point in X.

Proof. Define $\varphi\colon (0, \infty) \to (0, \infty)$ by $\varphi(t) = \tau > 0$ for all $t > 0$ in Theorem 3, the result follows. □

Corollary 7. *Let (X, G) be a complete G-metric space and $T\colon X \longrightarrow X$ be a continuous mapping. If there exists a constant $\tau > 0$, such that*

$$\tau + F(G(Tx, Ty, T^2y)) \leqslant F(\phi(G(x, y, Ty))),$$

$\forall\ x, y \in X$, $\phi \in \Phi$, $F \in \Delta_f$, then T has a unique fixed point in X.

Proof. Taking $\lambda_1 = 0$ and $\lambda_2 = 1$ for $q > 0$ in Corollary 6, the proof follows. □

In what follows, we construct an example to support the assumptions of Corollary 5.

Example 1. Let $X = [0, \infty)$ and define $G\colon X \times X \times X \to \mathbb{R}_+$ by $G(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$. Then (X, G) is a complete G-metric space. Take $\tau > 0$ and the mapping $T\colon X \longrightarrow X$ by:

$$Tx = \begin{cases} x^2 e^{-\tau}, & \text{if} \quad x \in [0,1]; \\ e^{-\tau}, & \text{if} \quad x > 1 \end{cases}$$

for all $x \in X$. Then T is continuous for all $x \in X$. Let $F(t) = \ln(t^2 + t)$, $t > 0$; then $F \in \Delta_f$. Observe that for all $x, y \in (1, \infty)$, there is nothing to show. Therefore, for all $x, y \in [0, 1]$, let $\lambda_1 = 0$ and $\lambda_2 = 1$. In order to show that the mapping T is a Jaggi-type hybrid (G, φ, F)-contraction, examine the following two cases:

Case 1: For some $q > 0$, take $q = 1$. Then

$$G(Tx, Ty, T^2y) = |Tx - Ty| + \left|Tx - T^2y\right| + \left|Ty - T^2y\right| =$$

$$
\begin{aligned}
&= \left|x^2e^{-\tau} - y^2e^{-\tau}\right| + \left|x^2e^{-\tau} - y^2e^{-2\tau}\right| + \left|y^2e^{-\tau} - y^2e^{-2\tau}\right| = \\
&= e^{-\tau}\Big[\, \left|x^2 - y^2\right| + \left|x^2 - y^2e^{-\tau}\right| + \left|y^2 - y^2e^{-\tau}\right| \Big] = \\
&= e^{-\tau}(G(x,y,Ty)) \leqslant e^{-\tau}(\mathcal{R}_A^q(x,y,Ty)). \qquad (20)
\end{aligned}
$$

In a similar manner, we have

$$
\begin{aligned}
[G(Tx,Ty,T^2y)+1] &= \Big[e^{-\tau}\Big(|x^2 - y^2| + |x^2 - y^2e^{-\tau}| + |y^2 - y^2e^{-\tau}|\Big) + 1\Big] = \\
&= \big[e^{-\tau}(G(x,y,Ty)) + 1\big] \leqslant \big[e^{-\tau}(\mathcal{R}_A^q(x,y,Ty)) + 1\big].
\end{aligned}
$$

From (20) and (16), we obtain

$$
\begin{aligned}
G(Tx,Ty,T^2y)\cdot[G(Tx,Ty,T^2y)+1] &= \\
&= e^{-\tau}(G(x,y,Ty))\cdot\big[e^{-\tau}(G(x,y,Ty)) + 1\big] \leqslant \\
&\leqslant e^{-\tau}(\mathcal{R}_A^q(x,y,Ty))\big[e^{-\tau}(\mathcal{R}_A^q(x,y,Ty)\big) + 1\big] < \\
&< e^{-\tau}\big[(\mathcal{R}_A^q(x,y,Ty))\cdot\big(\mathcal{R}_A^q(x,y,Ty) + 1\big)\big].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\tau + F(G(Tx,Ty,T^2y)) &= \tau + \ln(G(Tx,Ty,T^2y)^2 + G(Tx,Ty,T^2y)) < \\
&< \tau + \ln\big[e^{-\tau}(\mathcal{R}_A^q(x,y,Ty)^2 + \mathcal{R}_A^q(x,y,Ty))\big] = \\
&= \ln\big[(\mathcal{R}_A^q(x,y,Ty))^2 + \mathcal{R}_A^q(x,y,Ty)\big] = \\
&= F(\mathcal{R}_A^q(x,y,Ty)).
\end{aligned}
$$

Case 2: Similarly, for the case of $q = 0$, take $\lambda_1 = 0$, $\lambda_2 = 1$, and $x = T^2y$ for all $x, y \in X$ and obtain

$$
G(Tx,Ty,T^2y) \leqslant e^{-\tau}(\mathcal{R}_A^q(x,y,Ty)). \qquad (21)
$$

Therefore, it follows that

$$
\begin{aligned}
[G(Tx,Ty,T^2y)+1]\cdot G(Tx,Ty,T^2y) &< \\
&< e^{-\tau}\big[\big(\mathcal{R}_A^q(x,y,Ty) + 1\big)\cdot(\mathcal{R}_A^q(x,y,Ty))\big].
\end{aligned}
$$

Likewise as in Case 1, the inequality (21) yields

$$
\tau + F(G(Tx,Ty,T^2y)) \leqslant F(\mathcal{R}_A^q(x,y,Ty)).
$$

In Figure 1, we illustrate the validity of contractive inequality (19) using Example 1. Figure 1 shows that the right-hand side (RHS) of the inequal-

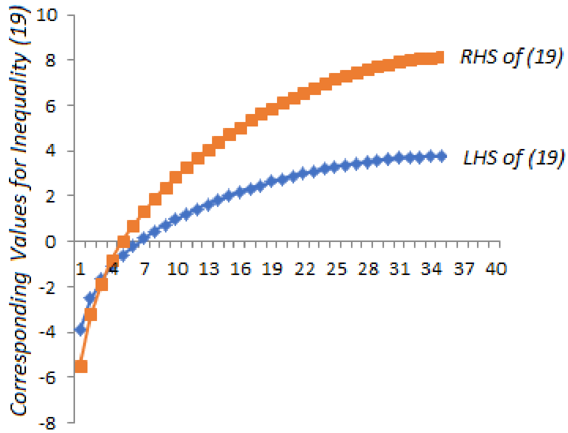

Figure 1: Illustration of contractive inequality (19) using Example 1

ity (19) dominates its left-hand side (LHS). Hence, all the assumptions of Corollary 5 are satisfied. Consequently, we see that $x = 0$ is the unique fixed point of T.

4. Application to solution of a nonlinear integral equation. Consider the following integral equation:

$$x(t) = q(t) + \int_0^t P(t,s)h(s,x(s))ds, \qquad t \in [0,k] = \mathcal{J}, \tag{22}$$

where $k > 0$, $h\colon \mathcal{J}\times\mathbb{R} \to \mathbb{R}$ and $q\colon \mathcal{J} \to \mathbb{R}$ are continuous, $P\colon \mathcal{J}\times\mathcal{J} \to \mathbb{R}_+$ is a function, and $P(s,\cdot) \in L^1(\mathcal{J})$ for all $t \in \mathcal{J}$.

Let $X = C(\mathcal{J},\mathbb{R}) = \{x\colon \mathcal{J} \to \mathbb{R}$ be continuous$\}$ and $G\colon X\times X\times X \to \mathbb{R}$ be a G-metric given by

$$G(x,y,z) = \sup_{t\in\mathcal{J}} \left\{ \left(|x(t)-y(t)| + |x(t)-z(t)| + |y(t)-z(t)|\right) e^{-\tau t} \right\}$$

for all $x,y,z \in X, t \in \mathcal{J}$. Clearly, (X,G) is a complete G-metric space. Define a mapping $T\colon X \longrightarrow X$ by

$$Tx(t) = q(t) + \int_0^t P(t,s)h(s,x(s))ds, \qquad t \in \mathcal{J} \text{ and } x \in X. \tag{23}$$

We know that a fixed point of T corresponds to a solution of (22). Now, we study the existence conditions of the integral equation (22) under the following assumptions.

Theorem 4. *Suppose that the following conditions are satisfied:*

(1) *there exists* $\tau > 0$*, such that for all* $s \in \mathcal{J}$ *and* $x, y \in X$:

$$|h(s, x(s)) - h(s, y(s))| \leqslant \frac{\tau}{\mu} e^{-\tau} |x(s) - y(s)|, \text{ where } \mu > 1;$$

(2) $\sup_{t \in \mathcal{J}} \int_0^t P(t, s) ds < 1.$

Then the integral equation given in (22) has a unique solution in X.

Proof. Let $x, y \in X$ be such that $Tx \neq Ty$. Then, employing (23) and the assumptions above, we have:

$$\begin{aligned} \Big|Tx(t) - Ty(t)\Big| &= \Big|\int_0^t P(t, s)h(s, x(s))ds - \int_0^t P(t, s)h(s, y(s))ds\Big| = \\ &= \Big|\int_0^t P(t, s)[h(s, x(s)) - h(s, y(s))]ds\Big| \leqslant \\ &\leqslant \int_0^t P(t, s) |h(s, x(s)) - h(s, y(s))| ds. \end{aligned}$$

Similarly, $|Tx(t) - T^2 y(t)| \leqslant \int_0^t P(t, s) |h(s, x(s)) - h(s, Ty(s))| ds$ and $|Ty(t) - T^2 y(t)| \leqslant \int_0^t P(t, s) |h(s, y(s)) - h(s, Ty(s))| ds$. Thus, it follows that

$$\begin{aligned} &G(Tx, Ty, T^2 y) = \\ &= \sup_{t \in \mathcal{J}} \left\{ \Big(|Tx(t) - Ty(t)| + |Tx(t) - T^2 y(t)| + |Ty(t) - T^2 y(t)| \Big) e^{-\tau t} \right\} \leqslant \\ &\leqslant \sup_{t \in \mathcal{J}} \left\{ \int_0^t P(t, s)\big[|h(s, x(s)) - h(s, y(s))| + |h(s, x(s)) - h(s, Ty(s))| + \right. \\ &\qquad \left. + |h(s, y(s)) - h(s, Ty(s))|\big] e^{-\tau s} ds \right\} \leqslant \end{aligned}$$

$$\leqslant \sup_{t\in\mathcal{J}}\left\{\int_0^t P(t,s)\big[\,|x(s)-y(s)|+|x(s)-Ty(s)|+\right.$$

$$\left.+\,|y(s)-Ty(s)|\big]\frac{\tau}{\mu}e^{-\tau}\cdot e^{-\tau s}ds\right\}\leqslant$$

$$\leqslant\left(\sup_{t\in\mathcal{J}}\int_0^t P(t,s)ds\right)\sup_{t\in\mathcal{J}}\left\{\int_0^t\big[\,|x(s)-y(s)|+|x(s)-Ty(s)|+\right.$$

$$\left.+\,|y(s)-Ty(s)|\big]\frac{\tau}{\mu}e^{-\tau}\cdot e^{-\tau s}ds\right\}\leqslant$$

$$\leqslant\frac{\tau}{\mu}e^{-\tau}\left[|x(t)-y(t)|+|x(t)-Ty(t)|+|y(t)-Ty(t)|\right]\cdot\int_0^t e^{-\tau s}ds=$$

$$=\frac{1}{\mu}e^{-\tau}\left[|x(t)-y(t)|+|x(t)-Ty(t)|+|y(t)-Ty(t)|\right]e^{-\tau t}.\quad(24)$$

Taking the supremum over all $t\in\mathcal{J}$ in (24) yields

$$G(Tx,Ty,T^2y)\leqslant\frac{1}{\mu}e^{-\tau}G(x,y,Ty)=\eta e^{-\tau}G(x,y,Ty),$$

where $\eta=\frac{1}{\mu}$, $\mu>1$. That is, $e^{\tau}G(Tx,Ty,T^2y)\leqslant\eta G(x,y,Ty)$. Taking the logarithm of both sides, we obtain

$$\ln(e^{\tau}G(Tx,Ty,T^2y))\leqslant\ln(\eta(G(x,y,Ty))),$$

which gives $\tau+\ln(G(Tx,Ty,T^2y))\leqslant\ln(\eta(G(x,y,Ty)))$. Defining $F\colon\mathbb{R}_+\to\mathbb{R}_+$ by $F(t)=\ln t$ for all $t>0$ and $\phi\colon\mathbb{R}_+\to\mathbb{R}_+$ by $\phi(t)=\eta t$, $t>0$, it follows that $\tau+F(G(Tx,Ty,T^2y))\leqslant F(\phi(G(x,y,Ty)))$.

Therefore, all the assumptions of Corollary 7 are satisfied. It follows that T has a unique fixed point in X, which corresponds to the unique solution of the integral equation (22). □

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