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J. Souissi

CHARACTERIZATION OF POLYNOMIALS VIA A RAISING OPERATOR

Abstract. This paper investigates a first-order linear differential operator $\mathcal{J}_\xi$, where $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \backslash (0,0)$, and $D := \frac{d}{dx}$. The operator is defined as $\mathcal{J}_\xi := x(xD + \mathbb{I}) + \xi_1 \mathbb{I} + \xi_2 D$, with $\mathbb{I}$ representing the identity on the space of polynomials with complex coefficients. The focus is on exploring the $\mathcal{J}_\xi$-classical orthogonal polynomials and analyzing properties of the resulting sequences. This work contributes to the understanding of these polynomials and their characteristics.

Key words: *orthogonal polynomials, classical polynomials, second-order differential equation, raising operator*

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1. Introduction. An orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is called *classical*, if $\{P'_n\}_{n\geqslant 0}$ is also orthogonal. This characterization is essentially the Hahn-Sonine characterization (see [9], [14]) of the classical orthogonal polynomials.

In a more general setting, let $\mathcal{O}$ be a linear operator acting on the space of polynomials, which sends polynomials of degree n to polynomials of degree $n + n_0$, where n_0 is a fixed integer ($n \geqslant 0$ if $n_0 \geqslant 0$ and $n \geqslant n_0$ if $n_0 < 0$). We call a sequence $\{p_n\}_{n\geqslant 0}$ of orthogonal polynomials $\mathcal{O}$-classical if $\{\mathcal{O}p_n\}_{n\geqslant 0}$ is also orthogonal.

In this paper, we consider the *raising operator*, $\mathcal{J}_\xi := x(xD + \mathbb{I}) + \xi_1 \mathbb{I} + \xi_2 D$, where $\xi = (\xi_1, \xi_2)$ is a nonzero free parameter and $\mathbb{I}$ represents the identity operator. We describe all the $\mathcal{J}_\xi$-classical orthogonal polynomial sequences.

The basic idea has been deduced by starting from the raising operator $\mathcal{U}_{\xi_2} := x(xD + \mathbb{I}) + \xi_2 D$ (see [1]). Now, to obtain a raising operator, we can add $\xi_1 \mathbb{I}$ to $\mathcal{U}_{\xi_2}$. Then we can consider the perturbed operator, given in the previous paragraph, $\mathcal{J}_\xi := \mathcal{U}_{\xi_2} + \xi_1\mathbb{I}$, where $(\xi_1, \xi_2) \neq (0,0)$ because the orthogonality is not preserved for $(\xi_1, \xi_2) = (0,0)$.

As a result associated to $\mathcal{U}_{\xi_2}$, we have that *the scaled Chebyshev polynomial sequence* $\{a^{-n}U_n(ax)\}_{n\geqslant 0}$ with $a^2 = -\xi_2^{-1}$ is the only $\mathcal{U}_{\xi_2}$-classical sequence, (for more details see [1]). In [2] the others prove that the scaled Bessel polynomial sequence $\{B_n^{(\frac{3}{2})}\}_{n\geqslant 0}$ is the only $\mathcal{J}_\xi$-classical orthogonal polynomial sequence for $\xi_2 = 0$. For the raising operator $\mathcal{J}_\xi$, the result is completely different. More precisely, in $\xi_1 \neq 0, \xi_2 \neq 0$ the Jacobi polynomial sequence $\{P_n^{(\alpha,\beta)}\}_{n\geqslant 0}$ is the only $\mathcal{J}_\xi$-classical orthogonal polynomial sequence with $\alpha = \frac{1-i\xi_1\mu}{2}, \beta = \frac{1+i\xi_1\mu}{2}$, $\mu^2 = \xi_2$, and $\xi_1\mu \neq i(2k+1)$, $k \in \mathbb{Z}_{\backslash\{-1,0\}}$.

The structure of the paper is the following: In Section 2, a basic background about forms, orthogonal polynomials is given. In Section 3, we find the $\mathcal{J}_\xi$-classical orthogonal polynomials. In Section 4, we give some properties of the sequence obtained.

2. Preliminaries. Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients. The algebraic dual space of $\mathbb{P}$ will be represented by $\mathbb{P}'$. We denote by $\langle u, p\rangle$ the action of $u \in \mathbb{P}'$ on $p \in \mathbb{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$, the sequence of moments of u with respect to the polynomial sequence $\{x^n\}_{n\geqslant 0}$.

Let us define the following operations in $\mathbb{P}'$. For linear functionals u, any polynomial g, and any $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, let $Du = u'$, gu, $\tau_{-b}u$ and $h_a u$ be the linear functionals defined by duality, [11]:

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad \langle gu, f\rangle := \langle u, gf\rangle, \ f \in \mathcal{P},$$

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \ \langle \tau_{-b}u, f\rangle := \langle u, \tau_b f\rangle = \langle u, f(x-b)\rangle, f \in \mathcal{P}.$$

A linear functional u is called *normalized* if it satisfies $(u)_0 = 1$.

Lemma 1. *[13], [11] For any $u \in \mathbb{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent:*

(i) $\langle u, P_{m-1}\rangle \neq 0, \langle u, P_n\rangle = 0, \ n \geqslant m.$

(ii) $\exists \lambda_k \in \mathbb{C}$, $0 \leqslant k \leqslant m-1$, $\lambda_{m-1} \neq 0$, *such that* $u = \sum_{k=0}^{m-1} \lambda_k u_k.$

As a consequence, the dual sequence $\{u_n^{[1]}\}_{n\geqslant 0}$ of $\{P_n^{[1]}\}_{n\geqslant 0}$, where $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x)$, $n \geqslant 0$, is given by

$$Du_n^{[1]} = -(n+1)u_{n+1}, \ n \geqslant 0.$$

Similarly, the dual sequence $\{\tilde{u}_n\}_{n\geqslant 0}$ of $\{\tilde{P}_n\}_{n\geqslant 0}$, where $\tilde{P}_n(x) := a^{-n}P_n(ax+ + b)$ with $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$, is given by

$$\tilde{u}_n = a^n(h_{a^{-1}} \circ \tau_{-b})u_n,\ n \geqslant 0.$$

The form u is called *regular* if we can associate with it a sequence $\{P_n\}_{n\geqslant 0}$, such that

$$\langle u, P_nP_m\rangle = r_n\delta_{n,m}, \quad n,\ m \geqslant 0, \quad r_n \neq 0, \quad n \geqslant 0.$$

The sequence $\{P_n\}_{n\geqslant 0}$ is then called a monic *orthogonal* polynomial sequence (MOPS) with respect to u. Note that $u = (u)_0u_0$, with $(u)_0 \neq 0$. When u is regular, let F be a polynomial, such that $Fu = 0$. Then $F = 0$ [11].

Proposition 1. *[11]. Let $\{P_n\}_{n\geqslant 0}$ be a MOPS with $\deg P_n = n$, $n \geqslant 0$, and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence. The following statements are equivalent:*

(i) *$\{P_n\}_{n\geqslant 0}$ is orthogonal with respect to u_0;*

(ii) *$u_n = \langle u_0, P_n^2\rangle^{-1}P_nu_0,\ n \geqslant 0$;*

(iii) *$\{P_n\}_{n\geqslant 0}$ satisfies the three-term recurrence relation*

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geqslant 0, \end{cases} \tag{1}$$

where

$$\beta_n = \langle u_0, xP_n^2\rangle\langle u_0, P_n^2\rangle^{-1},\ n \geqslant 0,$$
$$\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle\langle u_0, P_n^2\rangle^{-1} \neq 0,\ n \geqslant 0.$$

If $\{P_n\}_{n\geqslant 0}$ is a MOPS with respect to the regular form u_0, then $\{\tilde{P}_n\}_{n\geqslant 0}$ is a MOPS with respect to the regular form $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$, and satisfies [13]

$$\begin{cases} \tilde{P}_0(x) = 1, \quad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x),\ n \geqslant 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}(\beta_n - b)$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

An orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is called D-classical, if $\{P_n^{[1]}\}_{n\geqslant 0}$ is also orthogonal (*Hermite, Laguerre, Bessel or Jacobi*), [7], [9]. A second characterization of these polynomials, which will play the leading role in the sequel, is that they are the only polynomial solutions of the Second-Order Differential Equation (Bochner [5])

$$\text{(SODE)}: \quad \phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \lambda_n P_{n+1}(x), \ n \geqslant 0, \tag{2}$$

where ϕ, ψ are polynomials, ϕ monic, $\deg\phi = t \leqslant 2$, $\deg\psi = 1$, and $\lambda_n = (n+1)\big(\frac{1}{2}\phi''(0)n - \psi'(0)\big) \neq 0, \ n \geqslant 0$.
If $\{P_n\}_{n\geqslant 0}$ is a classical sequence satisfying (2), then $\{\tilde{P}_n\}_{n\geqslant 0}$ is also classical and satisfies (see [11])

$$\text{(SODE)}: \quad \tilde{\phi}(x)\tilde{P}''_{n+1}(x) - \tilde{\psi}(x)\tilde{P}'_{n+1}(x) = \lambda_n \tilde{P}_{n+1}(x), \ n \geqslant 0, \tag{3}$$

where $\tilde{\phi}(x) = a^{-t}\phi(ax+b)$ and $\tilde{\psi}(x) = a^{1-t}\psi(ax+b)$.

Now let us provide a summary of some basic characteristics of classical orthogonal polynomials. We focus on two families: the Bessel orthogonal polynomials (C1) and the Jacobi orthogonal polynomials (C2).

Bessel Orthogonal Polynomials (C1): For $n \geqslant 0$ and $\alpha \neq -\frac{n}{2}$, the Bessel orthogonal polynomials are denoted by $P_n(x) = B_n^{(\alpha)}(x)$, with $u_0 = \mathcal{B}^{(\alpha)}$. The coefficients are given by:

$$\beta_0 = -\frac{1}{\alpha}, \quad \beta_n = \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)}, \quad n \geqslant 0,$$
$$\gamma_n = -\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}, \quad n \geqslant 1.$$

The polynomials ϕ and ψ are x^2 and $-2(\alpha x+1)$, respectively, and λ_n are $(n+1)(n+2\alpha)$ for $n \geqslant 0$.

Jacobi Orthogonal Polynomials (C2): For $n \geqslant 0$ and $(\alpha, \beta \neq -n$, $\alpha+\beta \neq -n-1, n \geqslant 1)$, the Jacobi orthogonal polynomials are denoted by $P_n(x) = J_n^{(\alpha,\beta)}(x)$, with $u_0 = \mathcal{J}^{(\alpha,\beta)}$. The coefficients are given by:

$$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2}, \quad \beta_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$\gamma_n = \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \quad n \geqslant 1.$$

The polynomials ϕ and ψ are x^2-1 and $-(\alpha+\beta+2)x+\alpha-\beta$, respectively, and λ_n are $(n+1)(n+\alpha+\beta+2)$ for $n \geqslant 0$.

3. The $\mathcal{J}_\xi$-classical orthogonal polynomials. Recall the operator

$$\begin{aligned} \mathcal{J}_\xi : \mathbb{P} &\longrightarrow \mathbb{P} \\ f &\longmapsto \mathcal{J}_\xi(f) = (x^2+\xi_2)f' + (x+\xi_1)f. \end{aligned}$$

Definition 1. *We call a sequence $\{P_n\}_{n\geqslant 0}$ of orthogonal polynomials $\mathcal{J}_\xi$-classical if there exist a sequence $\{Q_n\}_{n\geqslant 0}$ of orthogonal polynomials, such that $\mathcal{J}_\xi P_n = Q_{n+1},\ n \geqslant 0$.*

For any MPS $\{P_n\}_{n\geqslant 0}$, we define the MPS $\{Q_n\}_{n\geqslant 0}$, given by $Q_{n+1}(x) := \frac{\mathcal{J}_\xi P_n(x)}{n+1}$, $n \geqslant 0$, or, equivalently,

$$(n+1)Q_{n+1}(x) := (x^2+\xi_2)P_n'(x) + (x+\xi_1)P_n(x),\ \ n \geqslant 0, \tag{4}$$

with the initial value $Q_0(x) = 1$.

Our next goal is to describe all the $\mathcal{J}_\xi$-classical polynomial sequences. Note that we need $\xi \neq 0$ to ensure that $\{Q_n\}_{n\geqslant 0}$ is an orthogonal sequence. Indeed, if we suppose that $\xi = (\xi_1,\xi_2) = 0$, the relation (4) becomes, for $x = 0$, $Q_{n+1}(0) = 0,\ n \geqslant 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geqslant 0}$.

The operator $\mathcal{J}_\xi$ raises the degree of any polynomial. Such operator is called *raising operator* [6, 10, 15]. By transposition of the operator $\mathcal{J}_\xi$, we get

$$^t\mathcal{J}_\xi = -\mathcal{J}_\xi + 2\xi_2\mathbb{I}. \tag{5}$$

Denote by $\{u_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ the dual basis in $\mathbb{P}'$ corresponding to $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. Then, according to Lemma 1 and (5), the relation

$$(x^2+\xi_2)v_{n+1}' + (x-\xi_1)v_{n+1} = -(n+1)u_n, \quad n \geqslant 0, \tag{6}$$

holds. Assume that $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x-\beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ \gamma_{n+1} \neq 0,\ n \geqslant 0, \end{cases} \tag{7}$$

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x-\rho_0, \\ Q_{n+2}(x) = (x-\rho_{n+1})Q_{n+1}(x) - \varrho_{n+1}Q_n(x),\ \varrho_{n+1} \neq 0,\ n \geqslant 0. \end{cases} \tag{8}$$

Next, the first result will be deduced as a consequence of the relations (4), (7), and (8).

Proposition 2. *The sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ satisfy the following finite-type relation:*

$$(x^2+\xi_2)P_n(x) = Q_{n+2}(x) + \theta_n Q_{n+1}(x) + \varpi_n Q_n(x),\ \ n \geqslant 0,$$

where

$$\begin{aligned}\theta_n &= (n+1)\big(\beta_n - \rho_{n+1}\big), \quad n \geqslant 0,\\ \varpi_n &= n\gamma_n - (n+1)\varrho_{n+1}, \quad n \geqslant 0,\end{aligned}$$

with the convention $\gamma_0 = 0$.

Proof. By differentiating (7), we obtain

$$P'_{n+2}(x) = (x-\beta_{n+1})P'_{n+1}(x) - \gamma_{n+1}P'_n(x) + P_{n+1}(x),\ \ n \geqslant 0.$$

Multiplying the last equation by $x^2+\xi_2$ and the relation (7) by $x+\xi_1$, take the sum of the two resulting equations, and substitute (4). Then we get

$$\begin{aligned}(n+3)Q_{n+3}(x) = (n+2)(x-\beta_{n+1})Q_{n+2}(x) - (n+1)\gamma_{n+1}Q_{n+1}(x)+\\ + (x^2+\xi_2)P_{n+1}(x), \quad n \geqslant 0.\end{aligned}$$

Using the three-term recurrence relation (8), we get

$$\begin{aligned}(x^2+\xi_2)P_{n+1}(x) = Q_{n+3}(x) + (n+2)\big(\beta_{n+1}-\rho_{n+2}\big)Q_{n+2}(x)+\\ + \big((n+1)\gamma_{n+1} - (n+2)\varrho_{n+2}\big)Q_{n+1}(x), \quad n \geqslant 0.\end{aligned}$$

In fact, this result is valid for $n+1$ replaced by n. More precisely, we have, for all $n \geqslant 0$,

$$\begin{aligned}&(x^2+\xi_2)P_n(x) =\\ &= Q_{n+2}(x) + (n+1)(\beta_n - \rho_{n+1})Q_{n+1}(x) + (n\gamma_n - (n+1)\varrho_{n+1})Q_n(x),\end{aligned}$$

with the convention $\gamma_0 = 0$. Hence the desired result. □

Note that, for $n = 0$, the Proposition 2 gives

$$Q_2(x) + (\beta_0 - \rho_1)Q_1(x) = x^2 + \xi_2 + \varrho_1,$$

and using the fact that $Q_1(x) = x + \xi_1$, we obtain

$$Q_2(x) = x^2 + (\xi_1 - \rho_1)x - \rho_1\xi_1 - \varrho_1. \tag{9}$$

By comparing (9) and (8) for $n = 0$, we obtain $\rho_1 = \frac{\beta_0+\xi_1}{2}$ and $\varrho_1 = -\frac{\xi_1^2+\xi_2}{2}$.

Now we establish, in the next lemma, an algebraic relation between the forms u_0 and v_0.

Lemma 2. *The forms u_0 and v_0 satisfy the following relation:*

$$(x^2 + \xi_2)v_0 = -\varrho_1 u_0.$$

Proof. According to Proposition 2, we obtain

$$\left\langle (x^2 + \xi_2)v_0, P_n \right\rangle = 0,\ n \geqslant 1. \tag{10}$$

On the other hand, by (9), we have $(x^2 + \xi_2) = Q_2 + (\beta_0 - \rho_1)Q_1 - \varrho_1$, and then

$$\left\langle (x^2 + \xi_2)v_0, P_0 \right\rangle = \left\langle v_0, Q_2 + (\beta_0 - \rho_1)Q_1 \right\rangle - \varrho_1 (v_0)_0 = -\varrho_1, \tag{11}$$

since $\{Q_n\}_{n\geqslant 0}$ is orthogonal with respect to the form v_0, where v_0 is supposed normalized. According to Lemma 1 and using (10) and (11), we obtain the desired result. □

It is clear that the formula (4) is a first-order differential equation satisfied by $\{P_n\}_{n\geqslant 0}$. Based on the last lemma, we obtain a first-order differential equation satisfied by $\{Q_n\}_{n\geqslant 0}$.

Proposition 3. *The following fundamental relation holds:*

$$Q'_{n+1}(x) = (n + 1)P_n(x),\ n \geqslant 0. \tag{12}$$

Proof. According to Proposition 1 (ii), the relation (6) can be written as follows:

$$(x^2 + \xi_2)[Q'_{n+1}(x)v_0 + Q_{n+1}(x)v'_0] + (x - \xi_1)Q_{n+1}v_0 = \lambda_n P_n(x)u_0,\ n \geqslant 0, \tag{13}$$

where $\lambda_n := -(n + 1)\langle v_0, Q^2_{n+1}\rangle\langle u_0, P^2_n\rangle^{-1},\ n \geqslant 0.$
Making $n = 0$ in (13), we get $(x^2 + \xi_2)v'_0 = (\xi_1 - x)v_0$. $(\lambda_0 = -\varrho_1)$.
Substituting this relation in (13), we obtain

$$(\lambda_n P_n - \varrho_1 Q'_{n+1})u_0 = 0.$$

Using the Lemma 2 and the fact that $\lambda_0 = -\varrho_1$ and taking into account regularity of u_0, we finally obtain $\lambda_0 Q'_{n+1}(x) = \lambda_n P_n(x),\ n \geqslant 0$. Comparing the degrees in the last equation, we get $\lambda_n = (n+1)\lambda_0,\ n \geqslant 0$, and, then, $Q'_{n+1}(x) = (n+1)P_n(x),\ n \geqslant 0$. □

According to Proposition 3, and using the Böchner characterization, we get the $\mathcal{J}_\xi$-classical orthogonal sequence. Now, we will describe all of the $\mathcal{J}_\xi$-classical polynomial sequences.

Theorem 1. *The $\mathcal{J}_\xi$-classical polynomial sequences are, up to a suitable affine transformation in the variable, one of the following D-classical polynomial sequences:*

(a) *if $\xi_1 = 0$, $P_n(x) = a^{-n}U_n(ax),\ n \geqslant 0$, with $a^2 = -\xi_2^{-1}$.*
(b) *if $\xi_2 = 0$, $P_n(x) = B_n^{(3/2)}(x)$, with $\xi_1 = 2$.*
(c) *if $\xi_1 \neq 0$ and $\xi_2 = -1$, $P_n(x) = P_n^{(\frac{1-\xi_1}{2},\frac{1+\xi_1}{2})}(x)$, with $\xi_1 \neq 2k+1$, $k \in \mathbb{Z}\backslash\{-1,0\}$.*
(d) *if $(\xi_1,\xi_2) \in \mathbb{C}^2_{\backslash\{(0,0)\}}$, $P_n(x) = P_n^{(\alpha,\beta)}(x)$, with $\alpha = \frac{1-i\xi_1\mu}{2}$, $\beta = \frac{1+i\xi_1\mu}{2}$, or $\mu^2 = \xi_2$,*
with $\xi_1\mu \neq i(2k+1),\ k \in \mathbb{Z}\backslash\{-1,0\}$.

Proof. Assume that $\{P_n\}_{n\geqslant 0}$ is a monic $\mathcal{J}_\xi$-classical orthogonal sequence. Then there exists a monic orthogonal sequence $\{Q_n\}_{n\geqslant 0}$ satisfying (4), which gives after differentiating and inserting (12), the following SODE:

$$(x^2+\xi_2)P''_{n+1}(x) + (3x+\xi_1)P'_{n+1}(x) = (n+1)(n+3)P_{n+1}(x),\ n \geqslant 0. \quad (14)$$

(a) if $\xi_1 = 0$, $P_n(x) = a^{-n}U_n(ax),\ n \geqslant 0$, with $a^2 = -\xi_2^{-1}$. (see [1])
(b) if $\xi_2 = 0$,

$$x^2 P''_{n+1}(x) - (-3x-\xi_1)P'_{n+1}(x) = (n+1)(n+3)P_{n+1}(x),\ n \geqslant 0.$$

According to Table C_1, $\{P_n\}_{n\geqslant 0}$ is the Bessel sequence of parameter α if $-2(\alpha x+1) = -3x-\xi_1$; in this case $\alpha = \frac{3}{2}$ and $\xi_1 = 2$.

(c) if $\xi_1 \neq 0$ and $\xi_2 = -1$,

$$(x^2-1)P''_{n+1}(x) + (3x+\xi_1)P'_{n+1}(x) = (n+1)(n+3)P_{n+1}(x),\ n \geqslant 0.$$

According to Table C_2, $\{P_n\}_{n\geqslant 0}$ is the Jacobi sequence of parameter (α,β) if $-(\alpha+\beta+2)x+\alpha-\beta = -3x-\xi_1$; in this case $\alpha = \frac{1-\xi_1}{2}$ and $\beta = \frac{1+\xi_1}{2}$, with $\xi_1 \neq 2k+1$, $k \in \mathbb{Z}\backslash\{-1,0\}$.

(d) if $(\xi_1, \xi_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$,

$$(x^2+\xi_2)P''_{n+1}(x)+(3x+\xi_1)P'_{n+1}(x) = (n+1)(n+3)P_{n+1}(x),\ n \geqslant 0.$$

According to Table C_2, $\{P_n\}_{n\geqslant 0}$ is the Jacobi sequence by a suitable affine transformation, $P_n(x) = \zeta^{-n}P_n^{(\alpha,\beta)}(\zeta x)$, with $\zeta^2 = -\xi_2^{-1}$, $\alpha = \frac{1-i\xi_1\mu}{2}$, $\beta = \frac{1+i\xi_1\mu}{2}$, or $\mu^2 = \xi_2$, with $\xi_1\mu \neq i(2k+1)$, $k \in \mathbb{Z}\backslash\{-1,0\}$.

□

4. Some properties of the sequence obtained. In the polynomial function space $\mathbb{P}$, we can introduce the linear operator, denoted here by $\mathbb{L}$:

$$\mathbb{L} := D.$$

Using (12), we obtain

$$\mathbb{L}(Q_{n+1}) = (n+1)P_n, \quad n \geqslant 0. \tag{15}$$

The operator $\mathbb{L}$ decreases the degree of a polynomial but preserves the orthogonality of the sequence $\{P_n\}_{n\geqslant 0}$.

We have the following result:

Theorem 2. *There exists a differential linear operator of order two $\mathcal{L}$, for which the polynomial $P_n(x)$, $n \geqslant 0$, is an eigenfunction. More precisely, we have:*

$$\mathcal{L}(P_n) = \theta_n P_n, \quad n \geqslant 0. \tag{16}$$

with $\theta_n = (n+1)^2$ as the corresponding eigenvalues, and where

$$\mathcal{L} := a_1(x)D^2 + a_2(x)D + a_3(x)\mathbb{I},$$

where

$$a_1(x) = x^2+\xi_2, \qquad a_2(x) = 3x+\xi_1, \qquad a_3(x) = 1.$$

Proof. Applying the $\mathcal{J}_\xi$ operator, and according to (4), we get

$$D \circ \mathcal{J}_\xi(P_n) = (n+1)^2 P_n, \quad n \geqslant 0.$$

This gives, after a simple calculation, the desired result. □

Note that, by applying the $\mathcal{L}$ operator to the X^n, $n \geqslant 0$, we obtain

$$\mathcal{L}(X^n) = \theta_n X^n + n\xi_1 X^{n-1} + n(n-1)\xi_2 X^{n-2},\ n \geqslant 0.$$

So, the matrix of the endomorphism $\mathcal{L}$ in the canonical basis $\{X^n\}_{n\geqslant 0}$ of $\mathbb{P}$ is given by

$$\mathbf{M}_{\mathcal{L}} = \begin{pmatrix} \theta_0 & \xi_1 & 2\xi_2 & 0 & \cdots & 0 \\ 0 & \theta_1 & 2\xi_1 & \ddots & \ddots & \vdots \\ & & \theta_2 & \ddots & n(n-1)\xi_2 & 0 \\ & & & \ddots & n\xi_1 & \ddots \\ & & & & \theta_n & \ddots \\ 0 & & & & & \ddots \end{pmatrix}.$$

Using the relation (16), we can write the matrix $\mathbf{M}_{\mathcal{L}}$ in the bases $\{P_n\}_{n\geqslant 0}$ as follows:

$$\mathbf{L} = \begin{pmatrix} \theta_0 & 0 & \cdots & \cdots & 0 \\ 0 & \theta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \theta_n & 0 \\ 0 & \cdots & \cdots & 0 & \ddots \end{pmatrix}.$$

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Jihad Souissi
Faculty of Sciences of Gabes
Department of Mathematics, Gabes University
Street Erriadh 6072 Gabes, Tunisia
E-mail: jihadsuissi@gmail.com & jihad.souissi@fsg.rnu.tn