DOI: 10.15393/j3.art.2024.14630

UDC 517.982.22, 517.521

E. Zikkos

# ON COMPLETE RIESZ–FISCHER SEQUENCES IN A HILBERT SPACE

**Abstract.** We prove that if $\{f_n\}_{n=1}^{\infty}$ is a complete Riesz–Fischer sequence in a separable Hilbert space $H$, then

$$T := \{f \in H\colon \sum |\langle f, f_n\rangle|^2 < \infty\}$$

is closed in $H$ if and only if $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz sequence. If the latter is also complete in $H$, then $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $H$.

**Key words:** *Riesz–Fischer sequences, Bessel sequences, Riesz sequences, Riesz bases, biorthogonal sequences, completeness*

**2020 Mathematical Subject Classification:** *42C15, 42C99*

**1. Introduction**. Let $H$ be a separable Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of vectors in $H$. We say that $\{f_n\}_{n=1}^{\infty}$ is a *Riesz basis* for $H$ if $f_n = V(e_n)$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $H$ and $V$ is a bounded bijective operator from $H$ onto $H$.

One of the many equivalences of Riesz bases [7, Theorem 1.1] states:

*A sequence is a Riesz basis for $H$, if and only if it is a complete Bessel sequence having a complete biorthogonal Bessel sequence in $H$.*

Recall that $\{f_n\}_{n=1}^{\infty}$ is a *Bessel* sequence if there is a positive constant $B$, so that

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant B\cdot \|f\|^2 \qquad \text{for all } f \in H,$$

(see [2, Definition 3.2.2]), or, equivalently, if

$$\sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 < \infty \qquad \text{for all } f \in H,$$

(see [3, Definition 7.1]). A sequence $\{f_n\}_{n=1}^{\infty}$ is *complete* if its closed span in $H$ is equal to $H$. Biorthogonality between two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ means that

$$\langle f_n, g_m \rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

Recently, Stoeva [7] has improved the above equivalence by assuming completeness of just one sequence.

**Theorem A.** [7, Theorem 2.5] *Let two sequences in $H$ be biorthogonal. If both of them are Bessel sequences and one of them is complete in $H$, then they are Riesz bases for $H$.*

**Remark 1**. *A short and elementary proof of this result has been given by the author in [10] basing on the notion of Riesz–Fischer sequences. We restate that proof here in Section 3.*

Following Young [9, Chapter 4, Section 2], $\{f_n\}_{n=1}^{\infty}$ is a *Riesz–Fischer* sequence in $H$ if the moment problem

$$\langle f, f_n \rangle = c_n$$

has at least one solution $f \in H$ for every sequence $\{c_n\}_{n=1}^{\infty}$ in the space $l^2(\mathbb{N})$.

Motivated by the result in [7], we now ask whether the following statement is true:

“Let two sequences in $H$ be biorthogonal. If both of them are Riesz–Fischer sequences and one of them is complete in $H$, then they are Riesz bases for $H$.”

Although we were not able to prove or disprove the statement above, we obtained another criterion for Riesz bases in Theorem 1, which is part of Theorem 2, where we explore the properties of complete Riesz–Fischer sequences in $H$.

**Theorem 1**. *If $\{f_n\}_{n=1}^{\infty}$ is a complete Riesz-Fischer sequence in $H$ having a biorthogonal sequence, which is complete in $H$, and $T := \{f \in H \colon \sum |\langle f, f_n \rangle|^2 < \infty\}$ is closed in $H$, then $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $H$.*

**2. Riesz-Fischer sequences, Bessel sequences, and Riesz sequences**. In [9, Chapter 4, Section 2, Theorem 3], we find the following two theorems that provide a necessary and sufficient condition that a sequence in $H$ is either a Riesz-Fischer sequence or a Bessel sequence. Both results are attributed to Nina Bari.

- $\{f_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence in $H$ if and only if there exists a positive number $A$ so that for any finite scalar sequence $\{\beta_n\}$ we have
$$A\sum|\beta_n|^2 \leqslant \|\sum \beta_n f_n\|^2. \tag{1}$$
- $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence in $H$ if and only if there exists a positive number $B$ so that for any finite scalar sequence $\{\beta_n\}$ we have
$$\|\sum \beta_n f_n\|^2 \leqslant B\sum|\beta_n|^2.$$

Now, it follows from (1) that a Riesz-Fischer sequence is also a minimal sequence, that is, each $f_n$ does not belong to the closed span of $\{f_k\}_{k\neq n}$ in $H$. Indeed, for fixed $n \in \mathbb{N}$ and any finite sum $\sum_{k\neq n}\beta_k f_k$, there is some $A > 0$, such that

$$\Big\|f_n + \sum_{k\neq n}\beta_k f_k\Big\|^2 \geqslant A(1 + \sum_{k\neq n}|\beta_k|^2) \geqslant A.$$

**Remark 2**. *It is well known that a sequence is minimal if and only if it has a biorthogonal sequence (see [2, Lemma 3.3.1]). A complete and minimal sequence, also called exact, has a unique biorthogonal sequence.*

It is clear now that a Riesz-Fischer sequence has at least one biorthogonal sequence. As stated in Casazza et al. [1], one of them is a Bessel sequence.

**Proposition A.** [1, Proposition 2.3, (ii)] *The Riesz-Fischer sequences in $H$ are precisely the families for which a biorthogonal Bessel sequence exists. In other words,*

*(Part a) Suppose that a Bessel sequence $\{f_n\}$ is biorthogonal to a sequence $\{g_n\}$ in $H$. Then $\{g_n\}$ is a Riesz–Fischer sequence.*

*(Part b) If $\{f_n\}$ is a Riesz-Fischer sequence, then it has a biorthogonal Bessel sequence.*

Another interesting result in Casazza et al. [1, Theorem 3.2] is as follows.

**Theorem B.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a complete Riesz–Fischer sequence in $H$. Then there is some $A > 0$, such that we have the following lower frame bound:*

$$A \cdot \|f\|^2 \leqslant \sum_{n=1}^{\infty}|\langle f, f_n\rangle|^2 \qquad \text{for all } f \in H.$$

**Remark 3.** *We point out that if a sequence $\{f_n\}_{n=1}^{\infty}$ in $H$ satisfies a lower frame bound*

$$A \cdot \|f\|^2 \leqslant \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \qquad \text{for all } f \in H, \tag{2}$$

*then it is a complete sequence in $H$. Indeed, if it were incomplete in $H$, a non-trivial function $f \in H$ would exist, such that $\langle f, f_n\rangle = 0$ for all $n \in \mathbb{N}$. But then this would contradict (2).*

And, finally, note that if $\{f_n\}_{n=1}^{\infty}$ in $H$ is both a Bessel sequence and a Riesz–Fischer sequence, then it is called a Riesz sequence (see Seip [6, Lemma 3.2]). That is, $\{f_n\}_{n=1}^{\infty}$ is a Riesz sequence if there are some positive constants $A$ and $B$, $A \leqslant B$, so that for any finite scalar sequence $\{\beta_n\}$ we have

$$A \sum |\beta_n|^2 \leqslant \| \sum \beta_n f_n\|^2 \leqslant B \sum |\beta_n|^2.$$

A Riesz sequence in $H$ is a Riesz basis for the closure of its linear span in $H$ (see [2, p. 68]). Therefore, a complete Riesz sequence in $H$ is a Riesz basis for $H$. Moreover, if $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $H$, it has a biorthogonal sequence $\{g_n\}_{n=1}^{\infty}$, which is also a Riesz basis for $H$ (see [2, Theorem 3.6.2]).

**3. Known results on Riesz bases and comparisons.** We have already mentioned the well known criterion:

"If $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are complete biorthogonal Bessel sequences in $H$, then they are Riesz bases for $H$."

In Theorem A, Stoeva concluded the same, by assuming completeness on just one sequence. We present now an alternative proof of her result. Consider the assumptions of Theorem A. From Proposition A (Part a), it follows that the two Bessel sequences that are biorthogonal to each other, are also Riesz–Fischer sequences, hence both are Riesz sequences. Completeness of one in $H$ means that it is a Riesz basis for $H$, thus having a unique dual Riesz basis and, clearly, this is the other biorthogonal sequence.

The following criterion is also known:

"Suppose that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are complete biorthogonal Riesz–Fischer sequences in $H$. Then they are Riesz bases for $H$."

Indeed, by Proposition A (Part b) and due to completeness, both families are Bessel sequences, hence Riesz bases for $H$.

**Remark 4.** *Suppose that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are two complete biorthogonal sequences in $H$, such that one is a Bessel sequence and the other one is a Riesz–Fischer sequence. Then they are not necessarily Riesz bases for $H$.*

As a counterexample, consider the exponential system

$$\{e^{i\lambda_n t}\}_{-\infty}^{\infty} \qquad \text{where} \qquad \lambda_n = \begin{cases} n+\frac{1}{4}, & n>0, \\ 0, & n=0, \\ n-\frac{1}{4}, & n<0. \end{cases}$$

The system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is minimal in $L^2(-\pi,\pi)$ [5, Theorem 5] and also complete [9, Chapter 3, Section 2, Theorem 4], (see also [4, Lecture 23, Problem 2]). But the frequencies $\lambda_n$ are uniformly separated, thus the system is a Bessel sequence in every $L^2(-a,a)$ space, $a>0$ [9, Chapter 4, Section 3, Theorem 4]. Since it is also complete and minimal in $L^2(-\pi,\pi)$, by Proposition A it has a unique biorthogonal sequence $\{g_n\}_{n\in\mathbb{Z}}$ in $L^2(-\pi,\pi)$, which is a Riesz–Fischer sequence. In addition, it follows from Young [8] that $\{g_n\}_{n\in\mathbb{Z}}$ is also complete in $L^2(-\pi,\pi)$. We point out, however, that the above exponential system is not a Riesz basis for $L^2(-\pi,\pi)$ [5, Theorem 4].

Everything said above motivate us to ask: "if two biorthogonal families in $H$ are Riesz–Fischer sequences and one of them is complete in $H$, are they Riesz bases for $H$?" At the moment this problem remains open.

## 4. Our result.

**Theorem 2.** *Let $H$ be a separable Hilbert space and let $\{f_n\}_{n=1}^{\infty}$ be a complete Riesz–Fischer sequence in $H$. Let*

$$T := \{f\in H\colon \sum |\langle f, f_n\rangle|^2 < \infty\}. \tag{3}$$

*Then $T$ is closed in $H$ if and only if $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz–Fischer sequence.*

*If either holds, then $T$ is equal to the closed span of the unique biorthogonal sequence to $\{f_n\}_{n=1}^{\infty}$, and there exist positive numbers $A$ and $B$, so that*

$$A\|f\|^2 \leqslant \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leqslant B\|f\|^2, \quad \text{for all } f\in T. \tag{4}$$

*If in addition $\{f_n\}_{n=1}^{\infty}$ has a complete biorthogonal sequence, then $\{f_n\}_{n=1}^{\infty}$ is a Riesz basis for $H$.*

**Proof.** Let $T^{\perp}$ be the orthogonal complement of $T$ in $H$. Suppose first that $T$ is closed in $H$. For each $n \in \mathbb{N}$, consider the unique decomposition $f_n = u_n + v_n$, where $u_n \in T$ and $v_n \in T^{\perp}$. Hence,

$$\langle f, f_n\rangle = \langle f, u_n\rangle \quad \text{for all } f \in T. \tag{5}$$

We will show below that $\{u_n\}_{n=1}^{\infty}$ is a Riesz basis for $T$.

First, from (5) and (3) we get

$$\sum |\langle f, u_n\rangle|^2 < \infty \quad \text{for all } f \in T.$$

Hence, $\{u_n\}_{n=1}^{\infty}$ is a Bessel sequence in $T$.

Second, by Theorem B and (5), there is some positive $A$, so that

$$A\|f\|^2 \leqslant \sum |\langle f, u_n\rangle|^2 \quad \text{for all } f \in T.$$

By Remark 3, this lower frame bound means that $\{u_n\}_{n=1}^{\infty}$ is complete in $T$.

Finally, let $\{c_n\} \in l^2$. Since $\{f_n\}_{n=1}^{\infty}$ is a Riesz–Fischer sequence, there exists some $F \in H$, such that $\langle F, f_n\rangle = c_n$. Clearly then $F$ belongs to $T$; thus $c_n = \langle F, f_n\rangle = \langle F, u_n\rangle$. This means that $\{u_n\}_{n=1}^{\infty}$ is a Riesz-Fischer sequence.

Therefore, we have proved that $\{u_n\}_{n=1}^{\infty}$ is a complete Bessel sequence and a Riesz–Fischer sequence simultaneously. Hence, $\{u_n\}_{n=1}^{\infty}$ is a Riesz basis for $T$.

Now it is clear that $\{u_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz basis $\{w_n\}_{n=1}^{\infty}$ for $T$. Therefore, $\langle u_n, w_n\rangle = 1$ and $\langle u_n, w_k\rangle = 0$ for $k \neq n$. Since $w_n \in T$ then

$$\langle f_n, w_n\rangle = 1 \quad \text{and} \quad \langle f_n, w_k\rangle = 0 \text{ for } k \neq n,$$

concluding that $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz sequence $\{w_n\}_{n=1}^{\infty}$ in $H$, which is of course a Riesz-Fischer one.

Next, we prove the converse statement, by supposing that $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz–Fischer sequence $\{g_n\}_{n=1}^{\infty}$ in $H$. By Proposition A (Part b) and since $\{f_n\}_{n=1}^{\infty}$ is complete in $H$, $\{g_n\}_{n=1}^{\infty}$ is also a Bessel sequence, hence a Riesz one as well. Thus, $\{g_n\}_{n=1}^{\infty}$ is a Riesz basis for $M := \overline{\text{span}}\{g_n\}_{n=1}^{\infty}$. It follows that $\{g_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz basis $\{h_n\}_{n=1}^{\infty}$ for $M$ and $M = \overline{\text{span}}\{h_n\}_{n=1}^{\infty}$.

We will show below that $M = T$.

First, let $M^{\perp}$ be the orthogonal complement of $M$ in $H$. Due to biorthogonality, for fixed $n \in \mathbb{N}$ we have $\langle f_n, g_n \rangle = 1$ and $\langle h_n, g_n \rangle = 1$. We also have $\langle f_n, g_m \rangle = 0$ and $\langle h_n, g_m \rangle = 0$ for all $m \neq n$. Therefore, $\langle (f_n - h_n), g_k \rangle = 0$ for all $k \in \mathbb{N}$, thus, $(f_n - h_n)$ belongs to $M^{\perp}$.

Note that since $\{h_n\}_{n=1}^{\infty}$ is a Riesz sequence in $H$, thus a Bessel one, then $\sum |\langle f, h_n \rangle|^2 < \infty$ for all $f \in H$. Since $(f_n - h_n) \in M^{\perp}$, for $f \in M$ we have $\langle f, f_n \rangle = \langle f, h_n \rangle$. Therefore,

$$\sum |\langle f, f_n \rangle|^2 < \infty \quad \text{for all } f \in M.$$

Thus, if $f \in M$ then $f$ belongs to $T$, hence

$$M \subset T. \tag{6}$$

Next, if $g \in T$, then $\sum |\langle g, f_n \rangle|^2 < \infty$, that is $\{\langle g, f_n \rangle\}$ belongs to the space $l^2$. Since $\{h_n\}_{n=1}^{\infty}$ is a Riesz basis for $M$, then $\{h_n\}_{n=1}^{\infty}$ is a Riesz–Fischer sequence, thus there exists a function $F \in M$, such that $\langle F, h_n \rangle = \langle g, f_n \rangle$. Since $F \in M$, then $\langle F, h_n \rangle = \langle F, f_n \rangle$. Thus, $\langle F, f_n \rangle = \langle g, f_n \rangle$ for all $n \in \mathbb{N}$. The completeness of $\{f_n\}_{n=1}^{\infty}$ in $H$ implies that $g = F$, thus $g$ belongs to $M$, hence

$$T \subset M. \tag{7}$$

From (6) and (7) we have $T = M$, thus $T$ is closed in $H$.

Therefore, we concluded that $T$ is closed in $H$ if and only if $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz-Fischer sequence.

Now, if either of the two assumptions holds, then $T = M$. Since $\{h_n\}_{n=1}^{\infty}$ is a Riesz basis for $M$ (similarly $\{u_n\}_{n=1}^{\infty}$ is a Riesz basis for $T$), then there exist positive numbers $A$ and $B$, such that one has the frame bounds (see [2, Proposition 3.6.4])

$$A\|f\|^2 \leqslant \sum_{n=1}^{\infty} |\langle f, h_n \rangle|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in M = T. \tag{8}$$

Replacing $\langle f, h_n \rangle$ in (8) by $\langle f, f_n \rangle$, which holds for $f \in M$, gives (4).

Finally, if either of the two assumptions holds, thus $\{f_n\}_{n=1}^{\infty}$ has a biorthogonal Riesz sequence, and in addition the latter is complete in $H$, then clearly this biorthogonal family and $\{f_n\}_{n=1}^{\infty}$ are Riesz bases for $H$.

The proof of Theorem 2 is now complete. □

**Acknowledgment.** I would like to thank very much the referee for all the comments and remarks that improved the quality of this article.

## References

[1] Casazza P., Christensen O., Li S., Lindner A. *Riesz-Fischer sequences and lower frame bounds.* Z. Anal. Anwendungen 2002, vol. 21, no. 2, pp. 305 – 314.

[2] Christensen O. *An introduction to Frames and Riesz Bases.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003. DOI: https://doi.org/10.1007/978-3-319-25613-9

[3] Heil C. *A basis theory primer.* Applied and Numerical Harmonic Analysis, Expanded edn. Birkhäuser/Springer. New York, 2011. DOI: https://doi.org/10.1007/978-0-8176-4687-5

[4] Levin B. Ya. *Lectures on Entire Functions.* Amer. Math. Soc., Providence, R. I., 1996.

[5] Redheffer R. M., Young R. M. *Completeness and Basis Properties of Complex Exponentials.* Trans. Amer. Math. Soc. 1983, vol. 277, no. 1, pp. 93 – 111.

[6] Seip K. *On the connection between exponential bases and certain related sequences in $L^2(-\pi,\pi)$.* J. Funct. Anal. 1995, vol. 130, no. 1, pp. 131 – 160.

[7] Stoeva D. T. *On a Characterization of Riesz Bases via Biorthogonal Sequences.* J. Fourier Anal. Appl. 2020, vol. 26, no. 4, article 67.

[8] Young R. M. *On complete biorthogonal systems.* Proc. Amer. Math. Soc. 1981, vol. 83, no. 3, pp. 537 – 540.

[9] Young R. M. *An introduction to Nonharmonic Fourier Series.* Revised first edition. Academic Press, Inc., San Diego, CA, 2001.

[10] Zikkos E. *Characterizing Riesz Bases via Biorthogonal Bessel sequences.* Carpathian Math. Publ. 2023, vol. 15, no. 2, pp. 377 – 380. DOI: https://doi.org/10.15330/cmp.15.2.377-380

*Received August 31, 2023.*
*In revised form, December 22, 2023.*
*Accepted December 23, 2023.*
*Published online January 19, 2024.*

Elias Zikkos
Department of Mathematics
Khalifa University
Abu Dhabi, United Arab Emirates
E-mail: elias.zikkos@ku.ac.ae