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SOME EMBEDDINGS RELATED TO HOMOGENEOUS TRIEBEL–LIZORKIN SPACES AND THE *BMO* FUNCTIONS

Abstract. As the homogeneous Triebel–Lizorkin space $\dot{F}^s_{p,q}$ and the space BMO are defined modulo polynomials and constants, respectively, we prove that BMO coincides with the realized space of $\dot{F}^0_{\infty,2}$ and cannot be directly identified with $\dot{F}^0_{\infty,2}$. In case $p < \infty$, we also prove that the realized space of $\dot{F}^{n/p}_{p,q}$ is strictly embedded into BMO. Then we deduce other results in this paper, that are extensions to homogeneous and inhomogeneous Besov spaces, $\dot{B}^s_{p,q}$ and $B^s_{p,q}$, respectively. We show embeddings between BMO and the classical Besov space $B^0_{\infty,\infty}$ in the first case and the realized spaces of $\dot{B}^0_{\infty,2}$ and $\dot{B}^0_{\infty,\infty}$ in the second one. On the other hand, as an application, we discuss the acting of the Riesz operator $\mathcal{I}_\beta$ on BMO space, where we obtain embeddings related to realized versions of $\dot{B}^\beta_{\infty,2}$ and $\dot{B}^\beta_{\infty,\infty}$.

Key words: *Besov spaces, BMO functions, realizations, Triebel–Lizorkin spaces*

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1. Introduction and the main result. The main result of this paper is the embeddings between the bounded mean oscillation space BMO and the homogeneous Triebel–Lizorkin spaces $\dot{F}^s_{p,q}$ in a certain sense. The spaces $\dot{F}^s_{p,q}$ defined by the Littlewood-Paley decomposition (abbreviated by LPd), in particular $\dot{F}^0_{\infty,2}$, as defined, e. g., in [10, (5.1)], are given by distributions modulo all polynomials; however, the space BMO is modulo constants, as defined, e. g., in [9]. We then observe that $\dot{F}^0_{\infty,2}$ *cannot be identified with* BMO, since for any polynomial f of degree $\geqslant 1$ it holds that $\|f\|_{\dot{F}^0_{\infty,2}} = 0$, while $\|f\|_{BMO} = \infty$. Concerning this identification, e. g., in [21, p. 243], the author replaced the space BMO-modulo constants by a space modulo polynomials (denoting it by $BMO^{(*)}$), which coincides with $\dot{F}^0_{\infty,2}$, we have $BMO \subsetneq BMO^{(*)}$; we refer, e. g., to the short comment at the end of [10, p. 70].

Another way to investigate the above identification is to introduce the realized space $\dot{\tilde{F}}^s_{p,q}$ (Definition 5 below) of $\dot{F}^s_{p,q}$, which is a subspace of $\mathcal{S}'_\nu$ (the collection of all tempered distributions modulo polynomials of degree $< \nu$) for some minimal values ν, which depend on $s - n/p$, see (2).

The concept of realization was introduced by G. Bourdaud in [4] for the homogeneous Besov spaces $\dot{B}^s_{p,q}$. This has an advantage in some fields, since there is no need to consider the spaces modulo polynomials. In general, realizations of homogeneous Besov and Triebel–Lizorkin spaces are defined up to a polynomial whose degree is less than an integer, denoted here by ν, which plays a crucial role in studying the convergence of the series associated to the LPd of such functions, see, e. g., [6], [7], see also the comment below at the beginning of Subsection 2.2.2 just after formula (2).

Nowadays, we know a lot of concrete characterizations on realized spaces (see e. g. Remarks 2 and 3 below), and there are many papers in this subject, e. g., [13], [14]. There are also various works related to the realizations of certain homogeneous spaces, as, e. g., in Navier-Stokes, Hardy, and Gagliardo-Nirenberg type estimates, pseudodifferential operators, pointwise multipliers and wavelets, see, e. g., [2], [12], [15], see also [1], in which further references on these topics may be found.

Thus, we show:

Theorem 1.

(i) *The identity* $BMO = \dot{\tilde{F}}^0_{\infty,2}$ *holds with equivalent seminorms.*

(ii) *If* $0 < p < \infty$, *then the embedding* $\dot{\tilde{F}}^{n/p}_{p,q} \hookrightarrow BMO$ *is proper.*

In relation with Theorem 1(i), the following statement (see, e. g., [5, Thm. VII.12, p. 147]) is proved:

Proposition 1. *A function* $f \in L_2^{\mathrm{loc}}$ *belongs to* BMO *if and only if*

(i) $$\int_{\mathbb{R}^n} \left(1 + |x|^{n+1}\right)^{-1} |f(x)|\, dx < \infty,$$

(ii) $$\sup_{y\in\mathbb{R}^n,\, k\in\mathbb{Z}} 2^{nk} \int_{|x-y|<2^{-k}} \sum_{j\geqslant k} |Q_j f(x)|^2\, dx < \infty.$$

The operators Q_j are defined in Subsection 2.1 below. This assertion gives a characterization of BMO; its proof has a certain history:

- The authors of [9, Thm. 3] have previously proved it by using the condition $\sup_{y\in\mathbb{R}^n,\, h>0} h^{-n} \int\limits_{|x-y|<h} \int\limits_0^h t|\nabla P_t f(x)|^2 dt\, dx < \infty$, where $P_t f$ is the Poisson integral of f, instead of (ii). Hence, it seems interesting to see if $\dot{\tilde{F}}^0_{\infty,2}$ can be endowed with seminorms defined by the Poisson semi-group (probably, open).
- In [20, Sect. 2], the author has considered (ii) in a continuous form, that is, the formula (1.1) in this reference. The same situation is given, e. g., in [18, IV.4.3, p. 159].

We turn to the embedding given in Theorem 1(ii); the authors in [19, Thm. 2(a)] proved: if $1 < p < \infty$ then

$$\|\mathcal{J}_{n/p} f\|_{BMO} \leqslant c \sup_{t>0} t \left|\{x \colon |f(x)| > t\}\right|^{1/p},$$

where $\mathcal{J}_{n/p}$ is the Bessel operator defined as $\mathcal{J}_s f := \mathcal{F}^{-1}((1+|\xi|^2)^{-s/2}\widehat{f}\,)$, $s \in \mathbb{R}$, and $|\{\ldots\}|$ denotes the Lebesgue measure of the set $\{\ldots\}$; the right-hand side can be easily estimated by $c\|f\|_p$. Then (which is well known)

$$H^{n/p}_p \hookrightarrow BMO \quad (1 < p < \infty), \tag{1}$$

where $H^{n/p}_p (1 < p < \infty)$ is the Bessel-potential space defined as the set of all functions f satisfying $\|f\|_{H^{n/p}_p} := \|\mathcal{J}_{-n/p} f\|_p < \infty$, but it is also well known that $H^{n/p}_p$ coincides with the inhomogeneous Triebel–Lizorkin space $F^{n/p}_{p,2}$; then the embedding properties of $F^s_{p,q}$ provide that $F^{n/p}_{p,q} \hookrightarrow BMO$ is satisfied for all $0 < p < \infty$ and all $0 < q \leqslant \infty$. Hence, dealing with $\dot{\tilde{F}}^{n/p}_{p,q}$ $(0 < p < \infty)$ presents the contribution of Theorem 1(ii), and now we can obtain (1) without using the operator $\mathcal{J}_s$, indeed we have $F^{n/p}_{p,q} \hookrightarrow \dot{\tilde{F}}^{n/p}_{p,q}$ since $F^{n/p}_{p,q} = L_p \cap \dot{F}^{n/p}_{p,q}$ (see [15, Prop. 2.5]).

The paper is organized as follows: In Section 2, we collect the useful tools, in particular some characterizations of the realized spaces. Section 3 is devoted to the proof of Theorem 1. In the last section, we discuss two corollaries (Subsection 4.1) of the main result for the inhomogeneous Besov spaces and their realized counterparts, and give some applications (Subsection 4.2) related to the actions of Riesz operator on BMO.

Notation. We denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We work in Euclidean space $\mathbb{R}^n$, then one writes $C^\infty(\mathbb{R}^n)$

as C^∞, $\mathcal{S}(\mathbb{R}^n)$ as $\mathcal{S}$, etc. For $s \in \mathbb{R}$, $[s]$ denotes its integer part. For $a \in \mathbb{R}$, we set $a_+ := \max(0, a)$. The symbol $\hookrightarrow$ means a continuous embedding. We denote by $P_{k,\mu}$ $(k \in \mathbb{Z}, \mu \in \mathbb{Z}^n)$ the dyadic cube $2^{-k}([0,1[^n + \mu)$. By $\|\cdot\|_p$ we denote the L_p quasi-norm. L_p^{loc} denotes the space of functions in $L_p(\Omega)$ for any compact set Ω in $\mathbb{R}^n$. $\mathcal{D}$ denotes the set of compactly supported functions in C^∞. The operators of translation τ_a $(a \in \mathbb{R}^n)$ and of dilation h_λ $(\lambda > 0)$ are defined by $\tau_a f := f(\cdot - a)$ and $h_\lambda f := f(\lambda^{-1} \cdot)$, respectively. For a measurable function f, $m_Q f := |Q|^{-1} \int_Q f(x)dx$ is its mean value over the set Q. For $f \in L_1$, the Fourier transform is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} f(x)\, dx,$$

the inverse by $\mathcal{F}^{-1}f(x) := (2\pi)^{-n}\widehat{f}(-x)$. The operator $\mathcal{F}$ can be extended to the space $\mathcal{S}'$ of tempered distributions in the usual way. For $m \in \mathbb{N}$, we denote by $\mathcal{P}_m$ the set of all polynomials in $\mathbb{R}^n$ of degree $< m$, e. g., $\mathcal{P}_1 := \{c \colon c \in \mathbb{C}\}$. We put $\mathcal{P}_0 := \{0\}$ and $\mathcal{P}_\infty$ the set of all polynomials in $\mathbb{R}^n$. For $m \in \mathbb{N}_0 \cup \{\infty\}$, the symbol $\mathcal{S}_m$ will be used for the set of functions $\varphi \in \mathcal{S}$ (the Schwartz space), such that $\langle u, \varphi \rangle = 0$ for all $u \in \mathcal{P}_m$, its topological dual is denoted by $\mathcal{S}'_m$. If $f \in \mathcal{S}'$, then $[f]_m$ denotes its equivalence class modulo $\mathcal{P}_m$. The constants $c, c_1, \dots$ are strictly positive, depend only on the fixed parameters as n, s, p, $q, \dots$ and some fixed functions, their values may change from one line to another.

Throughout the paper, the real numbers s, p, q satisfy $s \in \mathbb{R}$ and $p, q \in]0,\infty]$, unless otherwise stated.

2. Various function spaces.

2.1. Definition of Besov and Triebel–Lizorkin spaces. Throughout this work, we fix in C^∞ a radial function ρ, such that $0 \leqslant \rho \leqslant 1$, $\rho(\xi) = 1$ if $|\xi| \leqslant 1$ and $\rho(\xi) = 0$ if $|\xi| \geqslant 3/2$. We set $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$, which has support in the annulus $1/2 \leqslant |\xi| \leqslant 3/2$ and $\gamma(\xi) = 1$ in $3/4 \leqslant |\xi| \leqslant 1$. We define the operators S_j and Q_j $(\forall j \in \mathbb{Z})$ by $\widehat{S_j f}(\xi) := \rho(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{Q_j f}(\xi) := \gamma(2^{-j}\xi)\widehat{f}(\xi)$, which are defined on $\mathcal{S}'$, take values in the space of analytical functions of exponential type (see the Paley-Wiener theorem), and are uniformly bounded in $\mathcal{L}(L_p)$ $(1 \leqslant p \leqslant \infty)$ by virtue of the convolution Young inequality. We obtain the inhomogeneous LPd as follows:

For all $f \in \mathcal{S}$ (resp. $\mathcal{S}'$) and all $k \in \mathbb{Z}$, we have $f = S_k f + \sum_{j>k} Q_j f$ in $\mathcal{S}$

(resp. $\mathcal{S}'$). Recall that $\rho(2^{-k}\xi) + \sum_{j>k} \gamma(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n$.

We have the following definition:

Definition 1.

(i) *The inhomogeneous Besov space* $B^s_{p,q}$ *is the set of all* $f \in \mathcal{S}'$, *such that* $\|f\|_{B^s_{p,q}} := \|S_0 f\|_p + \big(\sum_{j\geqslant 1} (2^{js}\|Q_j f\|_p)^q\big)^{1/q} < \infty$.

(ii) *Let* $0 < p < \infty$. *The inhomogeneous Triebel–Lizorkin space* $F^s_{p,q}$ *is the set of all* $f \in \mathcal{S}'$, *such that*

$$\|f\|_{F^s_{p,q}} := \|S_0 f\|_p + \Big\| \Big(\sum_{j\geqslant 1} (2^{js}|Q_j f|)^q \Big)^{1/q} \Big\|_p < \infty.$$

To extend the definition to $\mathcal{S}'_\infty$, we use the following convention:

If $f \in \mathcal{S}'_\infty$, we define $Q_j f := Q_j f_1$ for any $f_1 \in \mathcal{S}'$, such that $[f_1]_\infty = f$.

Thus, Q_j are well-defined on $\mathcal{S}'_\infty$, since $Q_j f = 0$ $(\forall j \in \mathbb{Z})$ if and only if $f \in \mathcal{P}_\infty$ and:

For all $f \in \mathcal{S}_\infty$ (resp. $\mathcal{S}'_\infty$), we have $f = \sum_{j\in\mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ (resp. $\mathcal{S}'_\infty$).

Recall that $\sum_{j\in\mathbb{Z}} \gamma(2^j\xi) = 1$ for $\xi \neq 0$.

We get the following definition:

Definition 2.

(i) *The homogeneous Besov space* $\dot{B}^s_{p,q}$ *is the set of all* $f \in \mathcal{S}'_\infty$, *such that* $\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} (2^{js}\|Q_j f\|_p)^q\Big)^{1/q} < \infty$.

(ii) *Let* $0 < p < \infty$. *The homogeneous Triebel–Lizorkin space* $\dot{F}^s_{p,q}$ *is the set of all* $f \in \mathcal{S}'_\infty$, *such that* $\|f\|_{\dot{F}^s_{p,q}} := \Big\| \Big(\sum_{j\in\mathbb{Z}} (2^{js}|Q_j f|)^q \Big)^{1/q} \Big\|_p < \infty$.

(iii) *Let* $0 < q < \infty$. *The homogeneous space* $\dot{F}^s_{\infty,q}$ *is the set of all* $f \in \mathcal{S}'_\infty$, *such that*

$$\|f\|_{\dot{F}^s_{\infty,q}} := \sup_{k\in\mathbb{Z},\, \mu\in\mathbb{Z}^n} \Big(2^{kn} \int_{P_{k,\mu}} \sum_{j\geqslant k} (2^{js}|Q_j f(x)|)^q \, dx \Big)^{1/q} < \infty.$$

(iv) *For* $q=\infty$, *we set* $\dot F^s_{\infty,\infty} = \dot B^s_{\infty,\infty}$.

The spaces B, F (resp. $\dot B$, $\dot F$) are quasi-Banach for the above defined quasi-norms (resp. quasi-seminorms). Their definitions are independent of the choice of ρ, see, e. g., [16], [21, Sect. 2.3] and [10, Coro. 5.3]. In the above definitions, we can, also, change $Q_j f$ by $v_j * f$, where $v_j := 2^{jn}\mathcal{F}^{-1}v(2^j(\cdot))$ with $v \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$ be such that $v \geqslant 0$ and for $b > 2a > 0$, $v(\xi) \geqslant c > 0$ if $a \leqslant |\xi| \leqslant b$; see [6, Lem. 2.1.2] and again page 46 and Corollary 5.3 of [10]. On the other hand, it holds that:

(P1) $\mathcal{S} \hookrightarrow B, F \hookrightarrow \mathcal{S}'$.

(P2) $\mathcal{S}_\infty \hookrightarrow \dot B, \dot F \hookrightarrow \mathcal{S}'_\infty$.

(P3) $\dot B^s_{p,\min(p,q)} \hookrightarrow \dot F^s_{p,q} \hookrightarrow \dot B^s_{p,\max(p,q)}$, with $0<p<\infty$.

(P4) if $0 < q, r \leqslant \infty$, $s_1 > s_2$, and $0 < p_1 < p_2 < \infty$ are such that $s_1 - n/p_1 = s_2 - n/p_2$ then $\dot B^{s_1}_{p_1,q} \hookrightarrow \dot B^{s_2}_{p_2,q} \hookrightarrow \dot B^{s_2-n/p_2}_{\infty,q}$, $\dot F^{s_1}_{p_1,q} \hookrightarrow \dot B^{s_2}_{p_2,p_1}$ and $\dot F^{s_1}_{p_1,q} \hookrightarrow \dot F^{s_2}_{p_2,r}$, see [11].

(P5) $\lambda^{s-n/p}\|h_\lambda f\|_{\dot F^s_{p,q}} \sim \|f\|_{\dot F^s_{p,q}}$ for all $f \in \dot F^s_{p,q}$ and all $\lambda > 0$, if $p < \infty$; see, e. g., [21, Rem. 5.1.3/4], in case $p=\infty$ see [1]. The same holds when $\dot F$ is replaced with $\dot B$.

(P6) $\dot F^s_{\infty,q} \hookrightarrow \dot B^s_{\infty,\infty}$, see [1, Lem. 3]. Also, using the homogeneous Triebel–Lizorkin-type space $\dot F^{s,\tau}_{p,q}$, the set of all $f \in \mathcal{S}'_\infty$, such that

$$\|f\|_{\dot F^{s,\tau}_{p,q}} := \sup_{k\in\mathbb{Z}} \sup_{\mu\in\mathbb{Z}^n} 2^{kn\tau} \Big\| \Big(\sum_{j\geqslant k} (2^{js}|Q_j f|)^q \Big)^{1/q} \Big\|_{L_p(P_{k,\mu})} < \infty,$$

where $0<p<\infty$ and $0 \leqslant \tau < \infty$, see [22], we have $\dot F^s_{\infty,q} = \dot F^{s,1/p}_{p,q}$ and $\dot F^{s,\tau}_{p,q} \hookrightarrow \dot B^{s+n\tau-n/p}_{\infty,\infty}$, see again [22, Prop. 4.1], then taking $\tau = 1/p$ in the last embedding, we obtain the desired assertion.

Proposition 2. *A member f of $\mathcal{S}'_\infty$ belongs to $\dot F^s_{p,q}$ if and only if its first-order derivatives $\partial_\ell f$, $\ell = 1,\dots,n$, belong to $\dot F^{s-1}_{p,q}$. Moreover, $\sum_{\ell=1}^n \|\partial_\ell f\|_{\dot F^{s-1}_{p,q}}$ is an equivalent quasi-seminorm in $\dot F^s_{p,q}$. The same holds when $\dot F$ is replaced with $\dot B$.*

Proof. See, e. g., [8, Prop. 5]. The same proof, given in [6, Prop. 2.1.1] for the case of $\dot B$, can be used to obtain the case of $\dot F$, and also with $p=\infty$, since Q_j can be written as $\sum_{\ell=1}^n 2^{-j} v_\ell(2^{-j}D) \circ \partial_\ell$, where $v_\ell \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$ depends on γ. □

For further properties of the spaces $\dot{B}$, $\dot{F}$, B, and F, we refer the readers to, e. g., [10], [16], [21]. We also refer to the survey [22] in which the homogeneous Besov and Triebel–Lizorkin type spaces, $\dot{B}^{s,\tau}_{p,q}$ and $\dot{F}^{s,\tau}_{p,q}$, are studied (see (P6)); these spaces coincide with $\dot{B}^{s}_{p,q}$ and $\dot{F}^{s}_{p,q}$, respectively, if $\tau = 0$.

2.2. The realized spaces.

2.2.1. Generalities on realizations. We introduce the following definition and some remarks with respect to [6], [7], [13], [14].

Definition 3. *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \{0, \ldots, m\}$. Let E be a vector subspace of $\mathcal{S}'_m$ endowed with a quasi-seminorm, such that $E \hookrightarrow \mathcal{S}'_m$ holds. A realization of E in $\mathcal{S}'_k$ is a continuous linear mapping*

$$\sigma : E \to \mathcal{S}'_k \quad \text{such that} \quad [\sigma(f)]_m = f \quad \text{for all} \quad f \in E.$$

The image set $\sigma(E)$ is called the realized space of E with respect to σ.

For every f in E, the element $\sigma(f)$ is the unique representative of f in $\sigma(E)$; consequently, σ is completely characterized by its range. We say that a realization σ of E commutes with translations (resp. dilations) if $\tau_a \circ \sigma = \sigma \circ \tau_a$, $a \in \mathbb{R}^n$, (resp. $h_\lambda \circ \sigma = \sigma \circ h_\lambda$, $\lambda > 0$); this goes if and only if the range of σ is translation (resp. dilation) invariant.

Remark 1. *A subspace E of $\mathcal{S}'_m$ has generally infinitely many realizations in $\mathcal{S}'_k$ if $k < m$, in the case of $k = m$ the identity is the unique realization. Of course, with additional conditions such as translations or dilations invariance, a realization of E in $\mathcal{S}'_k$ for $k < m$ has some chances to be unique.*

We have the phenomenon that if a realization is known, we obtain other (see e. g. [7]):

Proposition 3. *Let $\sigma_0 : E \to \mathcal{S}'_k$ be a realization. For any finite family $(\mathcal{L}_\alpha)_{k \leqslant |\alpha| < m}$ of continuous linear functionals on E, the following formula defines a realization of E in $\mathcal{S}'_k$:*

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{k \leqslant |\alpha| < m} \mathcal{L}_\alpha(f)\, x^\alpha.$$

Conversely, any realization of E in $\mathcal{S}'_k$ is given in such a way.

Further information on difficulties arising in the study of translation or dilation commuting realizations is in, e. g., [7], [13]; see also Remark 4 and Subsection 2.2.3 below, for the F and B spaces, respectively.

2.2.2. Realizations of Triebel–Lizorkin spaces. To define the realized space of $\dot{F}^s_{p,q}$, we first fix the following natural number: to any 3-tuple (n, s, p) we associate:

$$\nu := \begin{cases} ([s - n/p] + 1)_+, & \text{if} \quad s - n/p \notin \mathbb{N}_0 \quad \text{or} \quad 1 < p \leqslant \infty, \\ s - n/p, & \text{if} \quad s - n/p \in \mathbb{N}_0 \quad \text{and} \quad 0 < p \leqslant 1. \end{cases} \tag{2}$$

The number ν characterizes the degree of the polynomials that define the realizations. In other words, if $f \in \dot{F}^s_{p,q}$, then the series $\sum_{j \in \mathbb{Z}} Q_j f$ (the homogeneous LPd of f) converges in $\mathcal{S}'_\nu$, and there exist polynomials r_j in $\mathcal{P}_\nu$, such that

$$f = \sum_{j \in \mathbb{Z}} (Q_j f - r_j) \quad \text{in} \quad \mathcal{S}',$$

see Propositions 4–6 below. For example, if $\nu = 0$ in the case of either $(s < n/p)$ or $(s = n/p$ and $0 < p \leqslant 1)$, we have $r_j = 0$ (recall $\mathcal{P}_0 = \{0\}$) and $\mathcal{S}'$ coincides with $\mathcal{S}'_0$; this case has been studied in several places, e. g., [16, pp. 55-56], [7, Prop. 4.6], [13, Thm. 4.1 and Rem. 4.3]. If $\nu \geqslant 1$ $(\mathcal{P}_\nu \neq \{0\})$, there exists a function $f \in \dot{F}^s_{p,q}$, such that the series $\sum_{j \leqslant 0} Q_j f$ diverges in $\mathcal{S}'_{\nu-1}$, see [6, Prop. 2.2.1] in which the proof given in $\dot{B}^s_{p,q}$ can be adapted to $\dot{F}^s_{p,q}$.

Finally, as mentioned in the Introduction, the number ν plays an important role in this work, and we refer to [1], [6], [7] for more information on this.

Second, we recall the following notion:

Definition 4. *A distribution $f \in \mathcal{S}'$ vanishes at infinity if*

$$\lim_{\lambda \downarrow 0} h_\lambda f = 0 \quad \text{in} \quad \mathcal{S}'.$$

The set of all such distributions is denoted by $\tilde{C}_0$.

Here are two examples of such distributions:

(i) $f \in \tilde{C}_0$ if $f \in L_p$ $(1 \leqslant p < \infty)$;

(ii) $\partial_\ell f \in \tilde{C}_0$ $(\ell = 1, \ldots, n)$ if $f \in L_\infty$ or $f \in \tilde{C}_0$.

Before turning to some properties related to the number ν, due to technical reasons, we introduce the definition of the realized spaces of $\dot{F}^s_{p,q}$, which can be found in [7, p. 483/Step 2] and [14, Sect. 2.3] if $p < \infty$, and in [1, Def. 5] if $p = \infty$.

Definition 5. *The realized space $\dot{\tilde{F}}^s_{p,q}$ is the set of all $f \in \mathcal{S}'_\nu$, such that $[f]_\infty \in \dot{F}^s_{p,q}$ and $\partial^\alpha f \in \tilde{C}_0$ for all $|\alpha| = \nu$, where ν is defined in (2). This space is endowed with the quasi-seminorm $\|f\|_{\dot{\tilde{F}}^s_{p,q}} := \|[f]_\infty\|_{\dot{F}^s_{p,q}}$.*

Remark 2. *The connections with the Lebesgue and homogeneous Sobolev spaces help us to understand the realized spaces. In this context, we recall that:*

- $\dot{\tilde{F}}^0_{p,2} = L_p$ *for* $1 < p < \infty$, *see [13, Prop. 5.2],*
- $\dot{\tilde{F}}^m_{p,2} = L_r \cap \dot{W}^m_p$ *for* $1 < p < \infty$, $m = 1, 2, \ldots$ *and* $1/r := 1/p - m/n > 0$, *and where* $\dot{W}^m_p$ *is the homogeneous Sobolev space endowed with the seminorm* $\|f\|_{\dot{W}^m_p} := \sum_{|\alpha|=m} \|f^{(\alpha)}\|_p$, *see [7, Thm. 5.3].*

Remark 3. *For convenience, in studying some analysis problems, the realizations can overcome some difficulties. For example, the pointwise multipliers in homogeneous Besov and Triebel–Lizorkin spaces, $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, are not defined; for this reason, it is better to work with realized spaces, see, e.g., [3, Thms. 1 and 2].*

We have the following property (see e.g., [7, Prop. 4.6], [13] if $p < \infty$ and [1] if $p = \infty$):

Proposition 4. *If $f \in \dot{F}^s_{p,q}$, the series $\sum_{j\in\mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\nu$ to an element denoted by $\sigma(f)$. The mapping $\sigma : \dot{F}^s_{p,q} \to \mathcal{S}'_\nu$ defined in such a way is a translation and dilation commuting realization of $\dot{F}^s_{p,q}$ in $\mathcal{S}'_\nu$, satisfying $[\sigma(f)]_\infty = f$ and $\partial^\alpha \sigma(f) \in \tilde{C}_0$ for all $|\alpha| = \nu$.*

On the other hand, [7, Sect. 4.3] provides a construction of realizations of $\dot{F}^s_{p,q}$ in $\mathcal{S}'$ in case $p, q \geqslant 1$, which can be easily extended to $p, q > 0$, see [13, Thms. 4.1 and 4.5], see also the proof of Lemma 9 and Remark 5 in [1] for the case $p = \infty$. Namely:

Proposition 5. *For all $f \in \dot{F}^s_{p,q}$, define $\sigma_{i,\nu}(f)$ ($i = 1, 2, 3$) by the following formulas:*

$$\sigma_{1,0}(f) := \sum_{j\in\mathbb{Z}} Q_j f, \text{ if either } (s < n/p) \text{ or } (s = n/p \text{ and } 0 < p \leqslant 1), \text{ here } \nu = 0; \quad (3)$$

$$\sigma_{2,\nu}(f) := \sum_{j\in\mathbb{Z}} \Big(Q_j f - \sum_{|\alpha|<\nu} (Q_j f)^{(\alpha)}(0) x^\alpha/\alpha! \Big), \text{ if either } (s - n/p \in \mathbb{R}^+ \backslash \mathbb{N}_0)$$
$$\text{or } (s - n/p \in \mathbb{N} \text{ and } 0 < p \leqslant 1), \quad \text{here } \nu = 1, 2, \ldots; \quad (4)$$

$$\sigma_{3,\nu}(f) := \sum_{j\geqslant 1} Q_j f + \sum_{j\leqslant 0}\Big(Q_j f - \sum_{|\alpha|<\nu}(Q_j f)^{(\alpha)}(0)x^\alpha/\alpha!\Big),$$
$$\text{if } s-n/p\in\mathbb{N}_0 \text{ and } 1<p\leqslant\infty, \quad \text{here}\,\nu = 1,2,\dots. \qquad (5)$$

Then $\sigma_{i,\nu}$ is a realization of $\dot F^s_{p,q}$ in $\mathcal{S}'$, such that all above series converge in $\mathcal{S}'$, $\partial^\alpha\sigma_{i,\nu}(f)\in\tilde C_0$ $(\forall|\alpha|=\nu)$, $[\sigma_{i,\nu}(f)]_\infty = f$ in $\mathcal{S}'_\infty$ and $\|[\sigma_{i,\nu}(f)]_\infty\|_{\dot F^s_{p,q}} = \|f\|_{\dot F^s_{p,q}}$. In (5) we can replace $\sum_{j\geqslant 1}$ and $\sum_{j\leqslant 0}$ with $\sum_{j\geqslant m}$ and $\sum_{j\leqslant m-1}$ for any $m\in\mathbb{Z}$, respectively.

Proposition 6. *The set $\sigma_{i,\nu}(\dot F^s_{p,q})$ ($\sigma_{i,\nu}$ is defined in (3)–(5)) is the collection of all $f\in\mathcal{S}'$, such that $[f]_\infty\in\dot F^s_{p,q}$, $\partial^\alpha f\in\tilde C_0$ $(\forall|\alpha|=\nu)$, and one of the following three conditions holds:*

(a) *There is no supplementary condition if either $(s<n/p)$ or $(s=n/p$ and $0<p\leqslant 1)$.*
(b) *f is of class $C^{\nu-1}$ and $f^{(\beta)}(0)=0$ for $|\beta|\leqslant\nu-1$, if either $(s-n/p\in\mathbb{R}^+\backslash\mathbb{N}_0)$ or $(s-n/p\in\mathbb{N}$ and $0<p\leqslant 1)$.*
(c) *f is of class $C^{\nu-1}$ and $f^{(\beta)}(0)=\sum_{j\geqslant 1}(Q_j f)^{(\beta)}(0)$ for $|\beta|\leqslant\nu-1$, if $s-n/p\in\mathbb{N}_0$ and $1<p\leqslant\infty$.*

The set $\sigma_{i,\nu}(\dot F^s_{p,q})$, also called the realized space of $\dot F^s_{p,q}$, is endowed with the same quasi-seminorm, i.e., $\|f\|_{\sigma_{i,\nu}(\dot F^s_{p,q})} := \|[f]_\infty\|_{\dot F^s_{p,q}}$. We also have $\sigma_{1,0}(\dot F^s_{p,q}) = \dot{\tilde F}^s_{p,q}$ and $\sigma_{i,\nu}(\dot F^s_{p,q})\subsetneqq\dot{\tilde F}^s_{p,q}$ $(i=2,3)$ if $\nu\geqslant 1$.

Proof. Denote by M the set of all $f\in\mathcal{S}'$ satisfying $[f]_\infty\in\dot F^s_{p,q}$, $\partial^\alpha f\in\tilde C_0$ $(\forall|\alpha|=\nu)$ and one of the conditions (a) or (b) or (c). By definition, we have the embedding $\sigma_{i,\nu}(\dot F^s_{p,q})\subset M$. Taking now $f\in M$, we have $f-\sigma_{i,\nu}([f]_\infty)\in\mathcal{P}_\infty$, and $\partial^\alpha(f-\sigma_{i,\nu}([f]_\infty))\in\tilde C_0$ if $|\alpha|=\nu$. But as $\tilde C_0\cap\mathcal{P}_\infty=\{0\}$, $f-\sigma_{i,\nu}([f]_\infty)=\sum_{|\beta|<\nu}a_\beta x^\beta$ (with $a_\beta=0$ if $\nu=0$), which implies $\partial^\beta(f-\sigma_{i,\nu}([f]_\infty))(0)=\beta!a_\beta$. Conditions (a)–(c) and (3)–(5) yield $a_\beta=0$ for all $|\beta|<\nu$, and, consequently, $f\in\sigma_{i,\nu}(\dot F^s_{p,q})$.

We now prove $\sigma_{i,\nu}(\dot F^s_{p,q})\subsetneqq\dot{\tilde F}^s_{p,q}$ $(i=2,3)$ if $\nu\geqslant 1$. The embedding follows from the Definition 5. To see that it is proper, let $f\in\sigma_{i,\nu}(\dot F^s_{p,q})$. Set $f_1(x):=f(x)+\sum_{|\beta|<\nu}x^\beta/\beta!$, then $f_1\in\dot{\tilde F}^s_{p,q}\backslash\sigma_{i,\nu}(\dot F^s_{p,q})$. Indeed, assume first that the condition (b) is satisfied; we have $f_1^{(\alpha)}(0)=1$ $(\forall|\alpha|<\nu)$.

Assume now that the condition (c) is satisfied; we have $Q_j f_1 = Q_j f$ $(\forall j \in \mathbb{Z})$, then $f_1^{(\alpha)}(0) = 1 + \sum_{j\geqslant 1}(Q_j f_1)^{(\alpha)}(0)$ $(\forall|\alpha| < \nu)$. The proof is complete. □

We add the following remark:

Remark 4. *In connection with Remark 1, we have the following assertions, where we principally refer to [7, Sect. 4] and [13, Sect. 3]:*

- *The mapping $\sigma_{1,0}$ defined in (3) commutes with translations, however, $\sigma_{i,\nu}$ $(i = 2, 3)$ defined in (4)–(5) are not.*
- *If $s - n/p \notin \mathbb{N}_0$, the mappings $\sigma_{1,0}$ and $\sigma_{2,\nu}$, defined in (3)–(4), commute with dilations; $\sigma_{1,0}(f)$ and $\sigma_{2,\nu}(f)$ are the unique representatives of functions from $\dot{F}^s_{p,q}$. If $s - n/p \in \mathbb{N}_0$ and $1 < p \leqslant \infty$, the mapping $\sigma_{3,\nu}$ defined in (5) does not commute with dilations.*
- *If $s - n/p \in \mathbb{N}_0$ and $0 < p \leqslant 1$, $\dot{F}^s_{p,q}$ has infinitely many dilation commuting realizations $\sigma := \tilde{\sigma} + \sum_{|\alpha|=\nu} \mathcal{L}_\alpha(\cdot)\, x^\alpha$, where $\tilde{\sigma} := \sigma_{1,0}$ or $\sigma_{2,\nu}$ (see (3)–(4)), and $(\mathcal{L}_\alpha)_{|\alpha|=\nu}$ is a family of continuous linear functionals on $\dot{F}^s_{p,q}$ satisfying $\mathcal{L}_\alpha \circ h_\lambda = \lambda^{-\nu}\mathcal{L}_\alpha$ $(\forall \lambda > 0)$.*

2.2.3. Realizations of Besov spaces. In the same way as in Definition 5, we define $\dot{\tilde{B}}^s_{p,q}$. Then we can take in Subsection 2.2.2 $\dot{B}$ and $\dot{\tilde{B}}$ instead of $\dot{F}$ and $\dot{\tilde{F}}$, respectively, by replacing in (2), Propositions 4–6, and Remark 4, the conditions $0 < p \leqslant 1$ and $1 < p \leqslant \infty$ with $0 < q \leqslant 1$ and $1 < q \leqslant \infty$, respectively.

2.2.4. Realized spaces and the integrability. We formulate some $\dot{\tilde{F}}^{n/p}_{p,q}$'s properties related to the integrability, which will be useful in what follows. Here $\nu = 0$ if $0 < p \leqslant 1$ and $\nu = 1$ if $1 < p \leqslant \infty$. For brevity, we set

$$\sigma_0(f) := \sigma_{1,0}(f),$$

and

$$\sigma_1(f) := \sigma_{3,1}(f) = \sum_{j\geqslant m} Q_j f + \sum_{j\leqslant m-1} \big(Q_j f - Q_j f(0)\big) \quad (m \in \mathbb{Z});$$

see (3) and (5), respectively.

We need to apply the following technical lemma (the Bernstein inequality) proved, e. g., in [21, Rem. 1.3.2/1].

Lemma 1. *Let $0 < p \leqslant q \leqslant \infty$ and $\alpha \in \mathbb{N}_0^n$. There exists a constant $c > 0$, such that*

$$\|f^{(\alpha)}\|_q \leqslant cR^{|\alpha|+n/p-n/q}\|f\|_p$$

holds, for all $R > 0$ and all f, such that $\mathrm{supp}\widehat{f} \subseteq \{\xi \colon |\xi| \leqslant R\}$.

Note that all following properties are also valid by changing $\dot{\tilde{F}}$ to $\dot{\tilde{B}}$ with the necessary modifications.

Proposition 7. *Let either $(0 < p < \infty)$ or $(p = \infty$ and $q = 2)$. Then it holds $\dot{\tilde{F}}^{n/p}_{p,q} \hookrightarrow L_1^{\mathrm{loc}} \cap B^0_{\infty,\infty}$. In the case $0 < p < \infty$, the embedding is proper.*

Proof.

Step 1: proof of the inclusion. Let $f \in \dot{\tilde{F}}^{n/p}_{p,q}$. We have $f - \sigma_\nu([f]_\infty) \in \mathcal{P}_\nu$, $(\nu = 0, 1)$. As $L_\infty \hookrightarrow L_1^{\mathrm{loc}} \cap B^0_{\infty,\infty}$ and $\mathcal{P}_\nu \hookrightarrow L_\infty$, it suffices to show $\sigma_\nu([f]_\infty) \in L_1^{\mathrm{loc}} \cap B^0_{\infty,\infty}$.

Substep 1.1: proof of $\sigma_\nu([f]_\infty) \in B^0_{\infty,\infty}$.

The case: $p \leqslant 1$. We split $\sigma_0([f]_\infty)$ as $g_1 + g_2$, where $g_1 := \sum_{j\geqslant 1} Q_j f$ and $g_2 := \sum_{j\leqslant 0} Q_j f$. Since $[f]_\infty \in \dot{F}^{n/p}_{p,q}$, $\dot{F}^{n/p}_{p,q} \hookrightarrow \dot{B}^0_{\infty,\infty}$ and $S_0 g_1 = S_0(Q_1 f)$, we have

$$\|S_0 g_1\|_\infty \leqslant \|\mathcal{F}^{-1}\rho\|_1 \|Q_1 f\|_\infty \leqslant c\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}}.$$

Also, as $\dot{F}^{n/p}_{p,q} \hookrightarrow \dot{B}^0_{\infty,p} \hookrightarrow \dot{B}^0_{\infty,1}$, we get

$$\|S_0 g_2\|_\infty \leqslant \|\mathcal{F}^{-1}\rho\|_1 \sum_{j\leqslant 0} \|Q_j f\|_\infty \leqslant c\|[f]_\infty\|_{\dot{B}^0_{\infty,1}}.$$

On the other hand, as $Q_k \sum_j Q_j = Q_k(Q_{k-1} + Q_k + Q_{k+1})$ since

$$Q_k Q_j = 0 \quad \text{if} \quad |k - j| \geqslant 2,$$

we have

$$\|Q_k(\sigma_0([f]_\infty))\|_\infty \leqslant c\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}} \quad (\forall k \in \mathbb{N}).$$

Thus, all these facts yield $\|\sigma_0([f]_\infty)\|_{B^0_{\infty,\infty}} < \infty$, see Definition 1(i).

The case: $1 < p < \infty$. We have $[\partial_\ell f]_\infty \in \dot{F}^{n/p-1}_{p,q} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$ $(\ell = 1, \ldots, n)$; here the integer associated with $\dot{F}^{n/p-1}_{p,q}$ is $\nu = 0$. Then we split $\sigma_0([\partial_\ell f]_\infty)$

as in the preceding decomposition g_1+g_2 by taking $\partial_\ell f$ instead of f. Thus, using Lemma 1, we obtain

$$\|S_0 g_1\|_\infty \leqslant \|\mathcal{F}^{-1}\rho\|_1 \|Q_1(\partial_\ell f)\|_\infty \leqslant c\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}},$$

$$\|S_0 g_2\|_\infty \leqslant \|\mathcal{F}^{-1}\rho\|_1 \sum_{j\leqslant 0} \|Q_j(\partial_\ell f)\|_\infty \leqslant c_1\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}} \sum_{j\leqslant 0} 2^j \leqslant c_2\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}}$$

and

$$2^{-k}\|Q_k(\sigma_0([\partial_\ell f]_\infty))\|_\infty \leqslant c\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}} \quad (\forall k \in \mathbb{N}).$$

Hence, $\sigma_0([\partial_\ell f]_\infty) \in B^{-1}_{\infty,\infty}$. We use the formula:

$$\partial_\ell \circ \sigma_\nu = \sigma_{(\nu-1)_+} \circ \partial_\ell \qquad (\ell = 1, \ldots, n),$$

proved in [7, Prop. 4.6]. We get $\partial_\ell \circ \sigma_1([f]_\infty) = \sigma_0([\partial_\ell f]_\infty) \in B^{-1}_{\infty,\infty}$, which implies $\sigma_1([f]_\infty) \in B^0_{\infty,\infty}$.

The case: $p = \infty$ and $q = 2$. We have $[\partial_\ell f]_\infty \in \dot{F}^{-1}_{\infty,2} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$ $(\ell = 1, \ldots, n)$, where the integer associated with $\dot{F}^{-1}_{\infty,2}$ is $\nu = 0$. Then we continue exactly as in the preceding case.

Substep 1.2: proof of $\sigma_\nu([f]_\infty) \in L_1^{\mathrm{loc}}$. The case $0 < p < \infty$ can be done as, e. g., in [2, pp. 29–30]. Then, we assume that $p = \infty$ and $q = 2$. We write $\sigma_1([f]_\infty) = g_3 + g_4$, where $g_3 := \sum_{j\geqslant m} Q_j f$ and $g_4 := \sum_{j\leqslant m-1} (Q_j f - Q_j f(0))$, where $m \in \mathbb{Z}$ is at our disposal, cf. Proposition 5. By Lemma 1, we first have

$$\begin{aligned}\|\nabla Q_j f\|_\infty \leqslant c_1 \sum_{\ell=0}^{n} \|\partial_\ell(Q_j f)\|_\infty \leqslant c_2 2^j \|Q_j f\|_\infty \leqslant \\ \leqslant c_2 2^j \|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}}, \quad (\forall j \in \mathbb{Z}).\end{aligned} \tag{6}$$

We choose $m \leqslant 0$, then clearly

$$\begin{aligned}|g_4(x)| \leqslant c_1|x| \sum_{j\leqslant m-1, m\leqslant 0} \|\nabla Q_j f\|_\infty \leqslant \\ \leqslant c_2|x|\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}} \sum_{j\leqslant 0} 2^j \leqslant c_3|x|\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}},\end{aligned}$$

hence $g_4 \in L_1^{\mathrm{loc}}$. We turn to g_3. Let Ω be a compact set in $\mathbb{R}^n$ and let x_0 a fixed point in Ω; there exists an integer $k := k(\Omega) < 0$, such that Ω is contained in the ball $\mathbb{B}(x_0, 2^{-k}) \subset \mathbb{R}^n$ centered at x_0 of radius 2^{-k}. Then we choose $m := k < 0$, and obtain

$$\int_\Omega |g_3(x)|\,dx \leqslant c\Big(\int_{\mathbb{B}(x_0,2^{-k})} \Big|\sum_{j\geqslant k} Q_j f(x)\Big|^2\,dx\Big)^{1/2}. \tag{7}$$

Now, since in the definition of $\dot{F}^0_{\infty,q}$ we can replace the dyadic cubes $P_{k,\mu}$ with the balls B_k in $\mathbb{R}^n$ of radius 2^{-k}, we continue as in [5, p. 153] (see also VII.13, p. 147 in this reference) to obtain that the right-hand side of (7) is bounded by $c\|[f]_\infty\|_{\dot{F}^0_{\infty,2}}$, where $c := c(\Omega) > 0$.

Step 2. To prove that the embedding is proper, it suffices to test the locally integrable function $u(x) := \mathrm{e}^{ix_1}$, $x := (x_1,\dots,x_n) \in \mathbb{R}^n$. Since $L_\infty \hookrightarrow B^0_{\infty,\infty}$, it holds $u \in B^0_{\infty,\infty}$. An easy computation gives $Q_k u = \gamma(2^{-k},0,\dots,0)u$ for all $k \in \mathbb{Z}$, then $Q_0 u = u$; recall that $\gamma(\xi) = 1$ in the annulus $3/4 \leqslant |\xi| \leqslant 1$. Now it is clear that $[u]_\infty \notin \dot{F}^{n/p}_{p,q}$ for any $0<p<\infty$. Indeed, if $[u]_\infty \in \dot{F}^{n/p}_{p,q}$ then $\|Q_0 u\|_p < \infty$, which is impossible. □

Remark 5. *We have $\dot{\tilde{F}}^0_{\infty,2} \hookrightarrow L_2^{\mathrm{loc}}$; the proof is similar to that given for Proposition 7. Also, concerning Proposition 7 in the case $p = \infty$ and $q = 2$, see Corollary 1 below.*

Proposition 8. *There exists a constant $c > 0$, such that the following estimate*

$$\frac{1}{|Q|}\int_Q |S_k f(x)|\,dx \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}$$

holds for all $f \in \dot{\tilde{F}}^{n/p}_{p,q}$, all $k \in \mathbb{Z}$, and all cubes Q in $\mathbb{R}^n$.

Proof. By Proposition 7, there exists a constant $c > 0$, such that it holds:

$$\int_{|x|\leqslant 1} |g(x)|\,dx \leqslant c\|[g]_\infty\|_{\dot{F}^{n/p}_{p,q}}, \quad \forall g \in \dot{\tilde{F}}^{n/p}_{p,q}.$$

Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Let $f \in \dot{\tilde{F}}^{n/p}_{p,q}$. Apply the last inequality to the function $g := f(r(\cdot)+x_0)$. By homogeneity (see (P5)) and translation invariance of $\|\cdot\|_{\dot{F}^{n/p}_{p,q}}$, it holds $\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}} = \|[f(r(\cdot)+x_0)]_\infty\|_{\dot{F}^{n/p}_{p,q}}$. Thus, we have the existence of a constant $c > 0$ independent of x_0, r, and f, such that

$$r^{-n}\int_{|x-x_0|\leqslant r} |f(x)|\,dx \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}. \tag{8}$$

We also have

$$r^{-n}\int\limits_{|x-x_0|\leqslant r}|S_kf(x)|\,dx\leqslant 2^{kn}\int\limits_{\mathbb{R}^n}|\mathcal{F}^{-1}\rho(2^ky)|\Big(r^{-n}\int\limits_{|u+y-x_0|\leqslant r}|f(u)|du\Big)dy. \quad (9)$$

Using (8), we see that the right-hand side of (9) is bounded by $c_1\|[f]_\infty\|_{\dot F^{n/p}_{p,q}}$, where $c_1 := c\|\mathcal{F}^{-1}\rho\|_1$. Let now Q be a cube in $\mathbb{R}^n$, and let B be the smallest ball in $\mathbb{R}^n$ containing Q; then $|B| = c|Q|$, where $c := c(n) > 0$. By the arbitrariness of x_0 and r, applying (9) with B, and using $|Q|^{-1}\int_Q \dots \leqslant$ $\leqslant c|B|^{-1}\int_B \dots$, the result follows. □

Proposition 9. *Let either $(0 < p < \infty)$ or $(p = \infty$ and $q = 2)$. If $f \in \dot{\tilde F}^{n/p}_{p,q}$, then it holds* $\displaystyle\int\limits_{\mathbb{R}^n}\big(1+|x|^{n+1}\big)^{-1}|f(x)|\,dx<\infty.$

Proof. Let $f \in \dot{\tilde F}^{n/p}_{p,q}$. As observed before, $f - \sigma_\nu([f]_\infty)$ is equal to 0 if $0 < p \leqslant 1$ and to a constant c if $1 < p \leqslant \infty$ (recall $\nu = 0, 1$), and since $\int\limits_{\mathbb{R}^n}(1+|x|^{n+1})^{-1}\,dx < \infty$, then it suffices to give a proof for functions $f \in \sigma_\nu(\dot F^s_{p,q})$.

Step 1: the case $p > 1$. As in the proof of Proposition 7 (Substep 1.2), f can be written as g_3+g_4, where $g_3 := \sum\limits_{j\geqslant m} Q_jf$ and $g_4 := \sum\limits_{j\leqslant m-1}(Q_jf-Q_jf(0))$, with $m \in \mathbb{Z}$ that will be chosen later. We set

$$U_i := \int\limits_{\mathbb{R}^n}\big(1+|x|^{n+1}\big)^{-1}|g_i(x)|\,dx \quad (i=3,4).$$

Estimate of U_3. Assume that $p < \infty$ and choose $m := 1$. We introduce p_1, such that $\max(1,p) < p_1 < \infty$, and set $p_1' := p_1/(p_1-1)$. By Hölder's inequality, it holds

$$U_3\leqslant\sum_{j\geqslant 1}\|Q_jf\|_{p_1}\Big(\int\limits_{\mathbb{R}^n}(1+|x|^{n+1})^{-p_1'}\,dx\Big)^{1/p_1'}\leqslant$$

$$\leqslant c_1\|[f]_\infty\|_{\dot B^{n/p_1}_{p_1,\infty}}\sum_{j\geqslant 1}2^{-jn/p_1}\leqslant c_2\|[f]_\infty\|_{\dot B^{n/p_1}_{p_1,\infty}};$$

we finish by using the embedding $\dot F^{n/p}_{p,q}\hookrightarrow\dot B^{n/p_1}_{p_1,\infty}$. Assume now $p=\infty$. We have

$$U_3\leqslant\sum_{\mu\in\mathbb{Z}^n}\Big(\int\limits_{P_{1,\mu}}\big(1+|x|^{n+1}\big)^{-2}\,dx\Big)^{1/2}\Big(\int\limits_{P_{1,\mu}}\Big|\sum_{j\geqslant m}Q_jf(x)\Big|^2\,dx\Big)^{1/2}. \quad (10)$$

We also have

$$\sup_{\mu\in\mathbb{Z}^n}\Big(\int\limits_{P_{1,\mu}}\Big|\sum_{j\geqslant m}Q_jf(x)\Big|^2\,dx\Big)^{1/2}\leqslant c\|[f]_\infty\|_{\dot F^0_{\infty,2}}, \tag{11}$$

which is proved in [5, p. 153]. Indeed, in this reference the author used the balls instead of dyadic cubes, but if $x\in P_{1,\mu}$, then $|x-2^{-1}\mu|\leqslant 2^{-1}\sqrt{n}<2^n$, hence $P_{1,\mu}$ is embedded in $\mathbb{B}(2^{-1}\mu,2^n)$: the ball in $\mathbb{R}^n$ centered at $2^{-1}\mu$ with radius 2^n. Then we choose $m:=-n$ and obtain

$$\int\limits_{P_{1,\mu}}\Big|\sum_{j\geqslant m}Q_jf(x)\Big|^2\,dx\leqslant\int\limits_{\mathbb{B}(2^{-1}\mu,\,2^{-(-n)})}\Big|\sum_{j\geqslant -n}Q_jf(x)\Big|^2\,dx;$$

cf. the formula (7) and the sentence just after. By inserting (11) into (10), and taking into account, first, that for $x\in P_{1,\mu}$, $1+|\mu|^{n+1}\leqslant c(1+|x|^{n+1})$ with a constant $c:=c(n)>0$ and, second, $\sum\limits_{\mu\in\mathbb{Z}^n}(1+|\mu|^{n+1})^{-1}<\infty$, we then get $U_3\leqslant c\|[f]_\infty\|_{\dot F^0_{\infty,2}}$.

Estimate of U_4. Let $0<b<1$. Using the estimate (6), then we have

$$\begin{aligned}|g_4(x)|&\leqslant 2^{1-b}\sum_{j\leqslant m-1}\|Q_jf\|_\infty^{1-b}\,|Q_jf(x)-Q_jf(0)|^b\leqslant\ (m:=1\text{ or }m:=-n)\\&\leqslant c_1|x|^b\sum_{j\leqslant 0}\|Q_jf\|_\infty^{1-b}\|\nabla Q_jf\|_\infty^b\leqslant\\&\leqslant c_2|x|^b\|[f]_\infty\|_{\dot B^0_{\infty,\infty}}\sum_{j\leqslant 0}2^{jb}\ \leqslant c_3|x|^b\|[f]_\infty\|_{\dot B^0_{\infty,\infty}}.\end{aligned}$$

Using the embedding $\dot F^{n/p}_{p,q}\hookrightarrow\dot B^0_{\infty,\infty}$ and $\int\limits_{\mathbb{R}^n}|x|^b(1+|x|^{n+1})^{-1}dx<\infty$, we finish.

Step 2: the case $0<p\leqslant 1$. Let p_2 be a number such that $1<p_2<\infty$. By the embedding $\dot F^{n/p}_{p,q}\hookrightarrow\dot F^{n/p_2}_{p_2,q}$, and the fact that $f\in\tilde C_0$ implies $\partial_\ell f\in\tilde C_0$ ($\ell=1,\dots,n$), we have $f\in\tilde{\dot F}^{n/p_2}_{p_2,q}$; recall that the integers associated with the spaces $\dot F^{n/p}_{p,q}$ and $\dot F^{n/p_2}_{p_2,q}$ are $\nu=0$ and $\nu=1$, respectively. Thus, an application of Step 1 with p_2 instead of p gives the result. □

2.3. Definition of the space *BMO*. The space BMO is the set of all $f\in L_1^{\mathrm{loc}}$ such that

$$\|f\|_{BMO}:=\sup_Q|Q|^{-1}\int\limits_Q|f(x)-m_Qf|\,dx<\infty,$$

where the supremum is taken over all finite cubes Q in $\mathbb{R}^n$; example of such a function: $x \mapsto \log|x|$. We note that in connection with the definition of the Triebel–Lizorkin space, we cannot identify BMO with the set of functions f such that $(\sum_{j\in\mathbb{Z}} |Q_j f|^2)^{1/2}$ is bounded, see again [10, p. 70] or [5, p. 154]. We also note that BMO has several properties, in particular:

Proposition 10. *Let $1 \leqslant p < \infty$. Every element f of BMO belongs to L_p^{loc} and satisfies (i) of Proposition 1.*

Proof. See [18, IV. 1.3, p. 144]. □

3. Proof of Theorem 1. *Step 1: proof of (i).* Let $f \in \dot{\tilde{F}}^0_{\infty,2}$. Remark 5 and Proposition 9 allow application of Proposition 1; then $f \in BMO$. Hence, the embedding in one direction is obtained. Conversely, let $f \in BMO$. By Propositions 1 and 10 we have $[f]_\infty \in \dot{F}^0_{\infty,2}$. We also have $f - \sigma_1([f]_\infty) \in \mathcal{P}_\infty$, where $\sigma_1 := \sigma_{3,1}$ (see (5)). By condition $\int_{\mathbb{R}^n} (1+|x|^{n+1})^{-1}|f(x)|\,dx < \infty$ and Proposition 9 applied to $\sigma_1([f]_\infty)$ (since $\sigma_1([f]_\infty) \in \dot{\tilde{F}}^0_{\infty,2}$), we obtain $f - \sigma_1([f]_\infty) \equiv c \in \mathbb{C}$; indeed, any non-constant polynomial G satisfies $\int_{\mathbb{R}^n} (1+|x|^{n+1})^{-1}|G(x)|\,dx = \infty$ (an easy exercise). The desired result follows.

Step 2: proof of (ii). Let us begin with some preparations. By the embedding $F^{n/p}_{p,q} \hookrightarrow F^{n/p_1}_{p_1,2}$ with $\max(p,1) < p_1 < \infty$, the estimate $\|f\|_{BMO} \leqslant c\|f\|_{F^{n/p_1}_{p_1,2}}$ (see (1) and the comment that follows just after) and the fact that $\|f\|_p + \|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}$ is an equivalent quasi-norm in $F^{n/p}_{p,q}$, see, e. g., [15, Prop. 2.5], for all $0 < p < \infty$ it holds

$$\|f\|_{BMO} \leqslant c(\|f\|_p + \|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}), \quad \forall f \in F^{n/p}_{p,q}.$$

In this inequality, replace f by $h_\lambda f$ for any $\lambda > 0$. Using the property (P5), the fact that $\|h_\lambda f\|_{BMO} = \|f\|_{BMO}$, and by letting $\lambda \to 0$, we get

$$\|f\|_{BMO} \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}, \quad \forall f \in F^{n/p}_{p,q}. \tag{12}$$

Now we turn to the embedding, and limit ourselves to $1 < p < \infty$. The case of $0 < p \leqslant 1$ can be obtained as in Step 2 of the proof of Proposition 9.

Take $f \in \dot{\tilde{F}}^{n/p}_{p,q}$ (recall that $\mathcal{P}_m \notin \dot{\tilde{F}}^{n/p}_{p,q}$ if $m \geqslant 2$) and set

$$f_k := \sum_{j \geqslant -k} Q_j f \quad \text{for} \quad k = 0, 1, \dots .$$

The sequence (f_k) has the following properties:

- by [8, Prop. 4] it holds that $\|[f_k]_\infty\|_{\dot{F}^{n/p}_{p,q}} \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}$ for all $k \in \mathbb{N}_0$,
- $f_k \in L_p$; indeed

$$\|f_k\|_p \leqslant \|[f]_\infty\|_{\dot{F}^{n/p}_{p,\infty}} \Big(\sum_{j \geqslant -k} 2^{-jn/p} \Big) \leqslant c\, 2^{kn/p} \|[f]_\infty\|_{\dot{F}^{n/p}_{p,\infty}} .$$

By applying (12) to f_k, we get

$$\|f_k\|_{BMO} \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}} , \quad \forall k \in \mathbb{N}_0 . \tag{13}$$

As $f = \sum_{j \in \mathbb{Z}} Q_j f$ in $\mathcal{S}'_\infty$, it holds that $f - f_k = \sum_{j < -k} Q_j f$ is in $\mathcal{S}'_\infty$. Then, by [8, Prop. 4] again, we obtain

$$\|[f - f_k]_\infty\|_{\dot{F}^{n/p}_{p,q}} = \Big\| \sum_{j < -k} Q_j f \Big\|_{\dot{F}^{n/p}_{p,q}} \leqslant c \Big\| \Big(\sum_{j < -k} (2^{jn/p} |Q_j f|)^q \Big)^{1/q} \Big\|_p , \quad \forall k \in \mathbb{N}_0 ,$$

with the change $\sup_{j < -k} 2^{jn/p} |Q_j f|$ in the inner norm $\|\cdot\|_p$ when $q = \infty$.

Set

$$v_k := \Big(\sum_{j < -k} (2^{jn/p} |Q_j f|)^q \Big)^{p/q}, \qquad k = 0, 1, \dots ,$$

(resp. taking $\sup_{j < -k} \dots$ if $q = \infty$). The positive sequence (v_k) satisfies $\lim_{k \to \infty} v_k = 0$ a. e. on $\mathbb{R}^n$, since

$$\Big(\sum_{j < -k} \big(2^{jn/p} |Q_j f(x)|\big)^q \Big)^{p/q} \leqslant \|[f]_\infty\|^p_{\dot{B}^0_{\infty,\infty}} \Big(\sum_{j < -k} 2^{jqn/p} \Big)^{p/q} \leqslant$$
$$\leqslant c 2^{-kn} \|[f]_\infty\|^p_{\dot{F}^{n/p}_{p,q}},$$

(resp. we have the bound $c\, 2^{-kn} \|[f]_\infty\|^p_{\dot{F}^{n/p}_{p,\infty}}$ if $q = \infty$). Also, as the assumption on f yields $v_k \leqslant \big(\sum_{j \in \mathbb{Z}} (2^{jn/p} |Q_j f|)^q \big)^{p/q} \in L_1$ (resp. also if $q = \infty$).

Thus, by dominated convergence theorem applied to the inequality

$$\|[f - f_k]_\infty\|^p_{\dot{F}^{n/p}_{p,q}} \leqslant c \int_{\mathbb{R}^n} v_k(x)\,dx,$$

we deduce that $\lim_{k\to\infty} \|[f - f_k]_\infty\|_{\dot{F}^{n/p}_{p,q}} = 0$, which yields

$$f_k \to f \quad \text{in} \quad \dot{\tilde{F}}^{n/p}_{p,q},$$

(see Definition 5). On the other hand, by Proposition 7, we get $f_k \to f$ in L_1^{loc}. Then, classically, there exists a subsequence (f_{k_j}) of (f_k), such that

$$f_{k_j} \to f \quad \text{a. e. on } \mathbb{R}^n \text{ as } j \to \infty.$$

Thus, for all finite cubes Q in $\mathbb{R}^n$, we have $f_{k_j} - m_Q f \to f - m_Q f$ a. e., which implies

$$\int_Q |f(x) - m_Q f|\,dx = \int_Q \lim_{j\to\infty} |f_{k_j}(x) - m_Q f|\,dx \leqslant$$
$$\leqslant \int_Q \lim_{j\to\infty} |f_{k_j}(x) - m_Q f_{k_j}|\,dx + |Q| \lim_{j\to\infty} |m_Q f - m_Q f_{k_j}|. \quad (14)$$

By the Fatou lemma and using (13) with (f_{k_j}), we get

$$\int_Q \liminf_{j\to\infty} |f_{k_j}(x) - m_Q f_{k_j}|\;dx \leqslant |Q| \liminf_{j\to\infty} \|f_{k_j}\|_{BMO} \leqslant$$
$$\leqslant c|Q|\,\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}. \quad (15)$$

For the last term in (14), it is clear that

$$|m_Q f - m_Q f_{k_j}| \leqslant |Q|^{-1} \int_Q |S_{-k_j-1} f(x)|\,dx,$$

which gives, in view of Proposition 8, $|m_Q f - m_Q f_{k_j}| \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}$ for all $j \in \mathbb{N}_0$ and all finite cubes Q in $\mathbb{R}^n$. Inserting both the last estimate and (15) into (14), dividing by $|Q|$ and taking the supremum over all Q, we get $\|f\|_{BMO} \leqslant c\|[f]_\infty\|_{\dot{F}^{n/p}_{p,q}}$; which is the desired result.

We prove that the embedding is proper. Let $f(x) := \log|x|$, $x \in \mathbb{R}^n$, and let $p_1 > p$. The function $\widehat{f}$ coincides with $c|\xi|^{-n}$ on $\mathbb{R}^n \backslash \{0\}$. Set $\widehat{\psi}(\xi) := |\xi|^{-n}\gamma(\xi)$, which satisfies $\psi \in \mathcal{S}_\infty$ and $\|Q_j f\|_{p_1} = 2^{-jn/p_1}\|\psi\|_{p_1}$. This implies $\|[f]_\infty\|_{\dot{B}^{n/p_1}_{p_1,p}} = \infty$ since $p < \infty$. But as $\dot{F}^{n/p}_{p,q} \hookrightarrow \dot{B}^{n/p_1}_{p_1,p}$, $0 < q \leqslant \infty$, we have $[f]_\infty \notin \dot{F}^{n/p}_{p,q}$. □

4. Some remarks.

4.1. *BMO* functions and the Besov spaces. An application of the main result yields the following assertions for the inhomogeneous Besov spaces and their homogeneous and realized counterparts.

Corollary 1. *The embedding $BMO \hookrightarrow B^0_{\infty,\infty}$ is proper.*

Proof. The inclusion follows by Proposition 7 and Theorem 1(i). To prove the embedding is proper, it suffices to observe that $B^0_{\infty,\infty} \nsubseteq L_1^{\mathrm{loc}}$. Recall that $B^0_{\infty,q} \subset L_1^{\mathrm{loc}}$ if and only if $0 < q \leqslant 2$, cf. [17, Thm. 3.3.2]. □

In [19, Thm. 2(b)], it was proved that if $s > 0$ then

$$\|\mathcal{J}_s f\|_{B^s_{\infty,\infty}} \leqslant c\|f\|_{BMO} \quad \text{for all} \quad f \in BMO \cap (L_1 + L_\infty),$$

$\mathcal{J}_s$ is defined in the Introduction; this implies $BMO \cap (L_1 + L_\infty) \subset B^0_{\infty,\infty}$ since $\mathcal{J}_s$ maps $B^0_{\infty,\infty}$ isomorphically onto $B^s_{\infty,\infty}$ and the expression $\|\mathcal{J}_s f\|_{B^s_{\infty,\infty}}$ is an equivalent to $\|f\|_{B^0_{\infty,\infty}}$, see e.g., [16, p. 67] or [21, Thm. 2.3.8]. Hence, the improvement given now by Corollary 1 *removes* the assumption $L_1 + L_\infty$ in this embedding.

Corollary 2. *It holds that $\dot{\tilde{B}}^0_{\infty,2} \hookrightarrow BMO \hookrightarrow \dot{\tilde{B}}^0_{\infty,\infty}$.*

Proof. This is immediate by $\dot{\tilde{B}}^0_{\infty,2} \hookrightarrow \dot{\tilde{F}}^0_{\infty,2} \hookrightarrow \dot{\tilde{B}}^0_{\infty,\infty}$, which can be easily obtained by $\dot{B}^0_{\infty,2} \hookrightarrow \dot{F}^0_{\infty,2}$, the property (P6) and the definition of ν, see (2). We note that the second embedding in this corollary is given in [12, Thm. 10.1]. □

In the same way, in [16, p. 169, Lines 2–3 p. 252] with a small modification, and in [5, p. 154], it was proved: *if $f \in BMO$, then $[f]_\infty \in \dot{B}^0_{\infty,\infty}$.* This can now be obtained easily by applying Theorem 1(i) and the property (P6).

4.2. Applications related to the Riesz operator. In order to investigate actions of the Riesz operator $\mathcal{I}_\beta$ (defined as $\mathcal{I}_\beta f := \mathcal{F}^{-1}(|\xi|^{-\beta}\widehat{f})$,

$\beta \in \mathbb{R}$) on BMO, we recall that $\mathcal{I}_\beta$ takes $\mathcal{S}_\infty$ to itself (an easy proof). It is therefore consistent to define $\mathcal{I}_\beta : \mathcal{S}'_\infty \to \mathcal{S}'_\infty$ as: if $f \in \mathcal{S}'_\infty$ then

$$\langle \mathcal{I}_\beta f, \varphi \rangle := \langle f_1, \mathcal{I}_\beta \varphi \rangle$$

for all $\varphi \in \mathcal{S}_\infty$ and all $f_1 \in \mathcal{S}'$, such that $[f_1]_\infty = f$.

4.2.1. The F-spaces. We have $\mathcal{I}_\beta$ maps $\dot{F}^s_{p,q}$ isomorphically onto $\dot{F}^{s+\beta}_{p,q}$ and $\|\mathcal{I}_\beta f\|_{\dot{F}^{s+\beta}_{p,q}} \sim \|f\|_{\dot{F}^s_{p,q}}$, see, e. g., [21, Thm. 5.2.3/1]. Then we get:

Proposition 11. *If $f \in \dot{\tilde{F}}^s_{p,q}$ then $\mathcal{I}_\beta([f]_\infty) \in \dot{F}^{s+\beta}_{p,q}$, and there exists a function $g \in \dot{\tilde{F}}^{s+\beta}_{p,q}$, such that $\mathcal{I}_\beta([f]_\infty) = [g]_\infty$ in $\mathcal{S}'_\infty$. In particular, if $f \in BMO$ then $\mathcal{I}_\beta([f]_\infty) \in \dot{F}^\beta_{\infty,2}$.*

Proof. It suffices to take $g := \sigma_{i,\nu}(\mathcal{I}_\beta([f]_\infty))$, where $\sigma_{i,\nu}$ is defined in Proposition 5. □

On the other hand, using the embedding $\dot{F}^\beta_{p,2} \hookrightarrow \dot{F}^0_{q,2}$ we obtain:

Proposition 12. *Let $0 < p < q < \infty$. Put $\beta := n/p - n/q$. If $f \in \dot{\tilde{F}}^0_{p,2}$, then there exists a function $g \in \dot{\tilde{F}}^0_{q,2}$, such that $\mathcal{I}_\beta([f]_\infty) = [g]_\infty$ in $\mathcal{S}'_\infty$ and $\|g\|_{\dot{\tilde{F}}^0_{q,2}} \leqslant c\|f\|_{\dot{\tilde{F}}^0_{p,2}}$. The positive constant c depends only on n, p, and q.*

Proof. The existence of g is obtained as in Proposition 11. For the estimate we apply the above embedding. □

Remark 6. *Clearly, the problem now arises: to see if $\mathcal{I}_\beta$ takes $\dot{\tilde{F}}^s_{p,q}$ to $\dot{\tilde{F}}^{s+\beta}_{p,q}$; it seems to be open.*

4.2.2. The B-spaces. An analogue of Proposition 11 holds in the case of Besov spaces. We have

$$\dot{B}^\beta_{\infty,2} \subseteq \mathcal{I}_\beta(BMO) \subseteq \dot{B}^\beta_{\infty,\infty} \qquad (\beta \in \mathbb{R}). \tag{16}$$

These inclusions are proved in [20, Thm. 3.4], at least for $\beta > 0$, with spaces $\dot{B}^s_{\infty,q}$ endowed with the functional

$$\mathcal{M}^{s,m}_q(f) := \Big(\int_{\mathbb{R}^n} \big(|h|^{-s} \|\Delta^m_h f\|_\infty\big)^q \frac{dh}{|h|^n} \Big)^{1/q},$$

for $m \in \mathbb{N}$ and $0 < s < m$, where Δ^m_h is the m-th difference operator ($\Delta^1_h f := \tau_{-h} f - f$ and $\Delta^m_h := \Delta^1_h \circ \Delta^{m-1}_h$, $m = 2, 3, \ldots$). Here the spaces

$\dot{B}^s_{\infty,q}$ are defined modulo polynomials of degree $\leqslant s$, cf. the last line in [20, p. 552]. Note that in $\dot{B}^s_{\infty,q}$ as defined by the LPd (cf. Definition 2), $\|\cdot\|_{\dot{B}^s_{\infty,q}}$ and $\mathcal{M}_q^{s,m}(\cdot)$ are *not equivalent*, since for any polynomial f of degree $\geqslant m$, $\|[f]_\infty\|_{\dot{B}^s_{\infty,q}} = 0$ while $\mathcal{M}_q^{s,m}(f) = \infty$ (e.g., $f(x) := x_1$ then $\Delta^1_h f(x) = h_1$). However, with $q = 2$ or ∞ and $s > 0$ we have:

(i) $\nu = [s] + 1$ for $\dot{B}^s_{\infty,q}$,

(ii) $\|.\|_{\dot{B}^s_{\infty,q}}$ and $\mathcal{M}_q^{s,m}(\cdot)$, with $m \geqslant \nu$, are equivalent in $\dot{\tilde{B}}^s_{\infty,q}$, see [14, Thm. 1.1].

In this sense, we can view (16) as follows:

- if $f \in BMO$, there exists a function $g \in \dot{\tilde{B}}^\beta_{\infty,\infty}$ such that $\mathcal{I}_\beta([f]_\infty) = [g]_\infty$ in $\mathcal{S}'_\infty$,
- if $f \in \dot{\tilde{B}}^\beta_{\infty,2}$, then $\mathcal{I}_{-\beta}([f]_\infty) \in \dot{B}^0_{\infty,2}$; as $\dot{B}^0_{\infty,2} \hookrightarrow \dot{F}^0_{\infty,2}$, by Proposition 11 we obtain the existence of a function $h \in BMO$, such that $\mathcal{I}_{-\beta}([f]_\infty) = [h]_\infty$ in $\mathcal{S}'_\infty$, this implies $[f]_\infty = \mathcal{I}_\beta([h]_\infty)$.

Remark 7. *As in Remark 6, we also have $\mathcal{I}_\beta : \dot{\tilde{B}}^s_{p,q} \to \dot{\tilde{B}}^{s+\beta}_{p,q}$ as an open question, cf. [15, Rem. 2.11].*

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