DOI: 10.15393/j3.art.2025.17250

UDC 517.58

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ON FUNCTIONAL INEQUALITIES FOR THE PSI FUNCTION

Abstract. Motivated by the works of Bougotta and Mercer, the authors in this paper study the monotonicity of the function $x \mapsto \psi(1+bx)^a/\psi(1+ax)^b$, and establish several inequalities involving the psi function.

Key words: *gamma function, polygamma function, inequalities*

2020 Mathematical Subject Classification: *33B15, 26D15*

1. Introduction. For Re, $x > 0$, we define the classical *gamma function* $\Gamma(x)$ and the *psi function* $\psi(x)$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. The psi function plays a significant role in various areas of mathematics, including number theory, special functions, and mathematical physics. It satisfies recurrence and reflection formulas, and it is used in the study of harmonic numbers and asymptotic expansions.

The recurrence relations of Γ and ψ are

$$\Gamma(1+x) = x\Gamma(x), \quad \psi(x+1) = \frac{1}{x} + \psi(x).$$

Note that

$$\psi(1) = -\gamma \quad \text{and} \quad \psi(1/2) = -2\log 2 - \gamma,$$

where γ is the *Euler-Mascheroni* constant. Throughout this paper, we denote by $c = 1.461632144968362\ldots$ the only positive root of the equation $\psi(x) = 0$ (see Eq. 6.3.19, [1]).

Due to the significance of the gamma function Γ, the psi function or digamma ψ and polygamma functions $\psi^{(n)}$, $n \in \mathbb{N}$ and their wide range of applications, numerous researchers have explored these functions extensively, deriving various two-sided inequalities and precise estimations. For example, the following remarkable inequalities involving these functions:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \quad 0<s<1,\, x>0,$$

$$\frac{xy}{x+y} < \frac{\ln(x+y)-\psi(x+y)}{(\ln x+\psi(x))(\ln y+\psi(y))}, \quad x,y \geqslant 0,$$

$$\ln\left(x+\frac{1}{2}\right)-\frac{1}{x} < \psi(x) < \ln(x+e^{\gamma})-\frac{1}{x}, \quad x>0,$$

$$1+z^2\psi'(z) < x^2\psi'(x)+y^2\psi'(y), \quad z^2=x^2+y^2,$$

appeared in [17], [4], [16], [3], respectively.

For further details, including inequalities, estimations, and applications of the gamma, psi, and polygamma functions, readers are referred to [4] – [8], [10] – [14], [20], [23] – [25] and the references therein.

In 2006, Alsina and Tomas [2] proved the following interesting inequality involving the gamma function:

$$\frac{1}{n!} \leqslant \frac{\Gamma(1+x)^n}{\Gamma(1+nx)}, \quad x\in[0,1],\, n=1,2,\ldots, \tag{1}$$

by using a geometrical method. Motivated by their result, J. Sándor [26] extended the inequality (1) as follows:

$$\frac{1}{\Gamma(1+a)} \leqslant \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}, \quad x\in[0,1],\, a\geqslant 1.$$

In [19], Mercer obtained the following inequalities:

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} < \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \quad a\in(0,1),$$

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} > \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \quad a\in\mathbb{R}\backslash(0,1),$$

with $0<x<y, 1+ax>0$, and $1+ay>0$. In 2006, Bougotta [13] proved the monotonicity property of $x\mapsto\Gamma(1+bx)^a/\Gamma(1+ax)^b$, by using the same method Sándor [26]. The inequalities of Sándor [26] and Mercer [19] follow from the inequality (2.4) of [21]. For the related inequalities of the gamma function, we refer the reader to [22].

Our first result is the counterpart of the above results and reads as follows.

Theorem 1. *For $a, b > 1$, the function*

$$f(x) = \frac{\psi(1+bx)^a}{\psi(1+ax)^b}$$

is increasing for $a > b$ and decreasing for $a < b$ in $(c-1, \infty)$, respectively.

In particular, for $1 < b < a$

$$\Big(\frac{\psi(1+bx)}{\psi(1+b(c-1))}\Big)^a \geqslant \Big(\frac{\psi(1+ax)}{\psi(1+a(c-1))}\Big)^b,$$

and the reverse inequality holds for $1 < a < b$.

Theorem 2. *The function $g(x) = 1/\psi(\cosh(x))$ is decreasing and convex from (c_1, ∞) onto $(1/\psi(\cosh(c_1)), 0)$, where $c_1 = \operatorname{arccosh}(c) = 0.92728\ldots$. In particular,*

$$\frac{2\psi(r)\psi(s)}{\psi((\sqrt{(1+r)(1+s)} + \sqrt{(r-1)(s-1)})/2)} \leqslant \psi(r) + \psi(s),$$

for all $r, s \in (c, \infty)$; equality holds for $r = s$.

2. Preliminaries and proofs.

The items of the following lemma will be used in our proofs; they can be found in [4], [14], [6], [15], [28], respectively.

Lemma 1. *For $x > 0$ we have*

1) $\log x - \dfrac{1}{x} < \psi(x) < \log x - \dfrac{1}{2x}$,

2) $\dfrac{1}{x} + \dfrac{1}{2x^2} < \psi'(x) < \dfrac{1}{x} + \dfrac{1}{x^2}$,

3) $\psi''(x) < \dfrac{1}{x} - 2\psi'(x)$,

4) $(\psi'(x))^2 + \psi''(x) > 0$,

5) $2\psi'(x) + x\psi''(x) < \dfrac{1}{x}$.

Lemma 2. *The function*

$$f(x) = \frac{\psi'(\cosh(x))\sinh(x)}{\psi(\cosh(x))^2},$$

is decreasing in $x \in (c_1, \infty)$.

Proof. Letting $r = \cosh(x)$,

$$f(x) = \frac{\sqrt{r^2-1}\,\psi'(r)}{(\psi(r))^2}.$$

Differentiating f with respect to x and using 4) and 5) of Lemma 1, we get

$$\begin{aligned} f'(x) &= \frac{-2(r^2-1)\psi'(r)^2 + \psi(r)\left(r\psi'(r) + (r^2-1)\,\psi''(r)\right)}{\psi^3(r)} \\ &< \frac{2(r^2-1)\psi''(r) + \psi(r)\left(r\psi'(r) + (r^2-1)\,\psi''(r)\right)}{\psi^3(r)} \\ &= \frac{(r^2-1)\psi''(r)(2+\psi(r)) + r\psi(r)\psi'(r)}{\psi^3(r)} \\ &< \frac{(r-1/r)(1/r-2\psi'(r))(2+\psi(r)) + r\psi(r)\psi'(r)}{\psi^3(r)} = \frac{f_1(r)}{\psi^3(r)}. \end{aligned}$$

Clearly $\psi^3(r)$ is positive because $r > c$. In order to show that f_1 is negative or, equivalently,

$$(r-1/r)(1/r-2\psi'(r))(2+\psi(r)) < -r\psi(r)\psi'(r),$$

it is enough to prove that

$$\frac{1}{r} - 2\psi'(r) < -\psi'(r), \tag{2}$$

and

$$(r-1/r)(2+\psi(r)) > r\psi(r). \tag{3}$$

The inequality (2) is valid and follows from Lemma 1 2). Also, the inequality (3) is equivalent to

$$f_1(r) = 2(r^2-1) - \psi(r) > 0\,. \tag{4}$$

The function f_1 is convex on $(0,\infty)$, because

$$\begin{aligned} f_1''(r) &= 4-\psi''(r) > 4-\left(\frac{1}{r}-2\psi'(r)\right) \\ &> 4-\frac{1}{r}+\frac{2}{r}+\frac{1}{r^2} > 0 \end{aligned}$$

by Lemma 1 3) & 2), and f_1 attains its minimum value 0.19997 at $r \approx 0.70407$. Hence the validity of inequality (4) follows immediately. This completes the proof of Lemma 2. □

Proof of Theorem 1. Let

$$g(x) = \log f(x) = a\log(\psi(1+bx)) - b\log(\psi(1+ax)).$$

Differentiating g with respect to x, we get

$$g'(x) = ab\left(\frac{\psi'(1+bx)}{\psi(1+bx)} - \frac{\psi'(1+ax)}{\psi(1+ax)}\right).$$

It is easy to see that the function $\psi'(z)/\psi(z)$ is positive and decreasing for $z \in (c,\infty)$. This implies that $g'(x)$ for $x > c-1$ is positive (negative) when $1 < b < a$ $(1 < a < b)$. The proof follows from this observation. □

Proof of Theorem 2. Differentiating g with respect to x, we get

$$g'(x) = -\frac{\psi'(\cosh(x))\sinh(x)}{\psi(\cosh(x))^2},$$

which is negative and increasing, hence g is convex. This implies that

$$\frac{1}{2}\left(\frac{1}{\psi(\cosh(x))} + \frac{1}{\psi(\cosh(y))}\right) \geqslant \frac{1}{\psi(\cosh(x+y)/2)}$$

$$= 1/\psi\left(\frac{\sqrt{(\cosh(x)+1)(\cosh(y)+1)}}{2} + \frac{\sqrt{(\cosh(x)-1)(\cosh(y)-1)}}{2}\right).$$

Setting $r = \cosh(x)$ and $s = \cosh(y)$, we complete the proof. □

For the formulation of the following result, we denote $z' = \sqrt{1-z^2}$, $z \in (0,1)$. The functional inequality of the following corollary is reminiscent of Theorem 5.12 of [9].

Corollary. *The following inequality:*

$$\psi(r)+\psi(s) \leqslant 2\psi\left(\sqrt{\frac{2rs}{1+rs+r's'}}\right)$$

holds for $r, s \in (0,1)$, *with equality for* $r = s$.

Proof. Let $h = \psi(1/\cosh(x))$ for $x > 0$. We get

$$h'(x) = -\psi'(1/\cosh(x))\frac{\tanh(x)}{\cosh(x)} = -\tanh(x)h_1(x).$$

Letting $u = 1/\cosh(x)$, one has

$$h_1'(x) = u'(\psi'(u) + u\psi''(u)),$$

which is positive, because $u' < 0$. By Lemma 1 2) & 3),

$$\begin{aligned}
\psi'(u) + u\psi''(u) &< \frac{1}{u} + \frac{1}{u^2} + u\Big(\frac{1}{u} - 2\psi'(u)\Big)\\
&< \frac{1}{u} + \frac{1}{u^2} + 1 - 2u\Big(\frac{1}{u} + \frac{1}{2u^2}\Big)\\
&= \frac{1}{u} + \frac{1}{u^2} + 1 - 2 - \frac{1}{u}\\
&= \frac{1}{u^2} - 1 < 0.
\end{aligned}$$

Thus, h_1 is increasing, and also $\tanh(x)$ is increasing. Clearly, h' is decreasing and negative, hence h is concave in $x > 0$. This implies

$$\psi(1/\cosh((x+y)/2)) \geqslant \frac{\psi(1/\cosh(x)) + \psi(1/\cosh(y))}{2}.$$

The desired inequality follows if we let $r = 1/\cosh(x), s = 1/\cosh(y)$ and use the identity $\cosh^2((x+y)/2) = (1 + xy + x'y')/(2xy)$. □

For convenience, we use the notation $\mathbb{R}_+ = (0, \infty)$.

Lemma 3. *[21, Thm 2.1] Let* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable, log-convex function, and let* $a \geqslant 1$. *Then* $g(x) = (f(x))^a/f(a\,x)$ *decreases in its domain. In particular, if* $0 \leqslant x \leqslant y$, *then the following inequalities*

$$\frac{(f(y))^a}{f(a\,y)} \leqslant \frac{(f(x))^a}{f(a\,x)} \leqslant (f(0))^{a-1}$$

hold. If $0 < a \leqslant 1$, *then the function* g *is an increasing function on* $\mathbb{R}_+$ *and inequalities are reversed.*

Corollary. *For* $k > 1$ *and* $c < x \leqslant y$, *the following inequality holds:*

$$\left(\frac{\psi(x)}{\psi(y)}\right)^k \leqslant \frac{\psi(kx)}{\psi(ky)}.$$

Proof. Let $g_1(x) = \log(1/\psi(x))$. Differentiating g_1 twice with respect to x, we get, by Lemma 1:

$$g_1''(x) = \frac{\psi'(x)^2 - \psi''(x)\psi(x)}{\psi(x)^2} > \frac{-\psi''(x)(1+\psi(x))}{\psi(x)^2} > 0,$$

which implies that g_1 is convex. Now the rest of proof follows easily from Lemma 3. □

Theorem 3. *The function* $f(x) = \operatorname{artanh}(\psi(\tanh(x)))$ *is strictly increasing and concave from* (c, ∞) *onto* (l, m)*, where*

$$f(c) = \operatorname{artanh}(\psi(\tanh(c))) = -0.9934\ldots = l$$

and

$$f(\infty) = -\operatorname{artanh}(\gamma) = -0.6582\ldots = m.$$

In particular,

1) $\psi\left(\frac{r+s}{1+rs+r's'}\right) > \frac{\psi(r)+\psi(s)}{1+\psi(r)\psi(s)+\sqrt{1-\psi(r)^2}\sqrt{1-\psi(s)^2}},$

   for all $r, s \in (0,1)$*, where* $r' = \sqrt{1-r^2}$ *and* $s' = \sqrt{1-s^2}$,

2) $\frac{1+\psi(\tanh(r))}{1-\psi(\tanh(r))}\frac{1-\psi(\tanh(s))}{1+\psi(\tanh(s))} < e^{2a(r-s)}$, $\forall r, s \in (c, \infty)$, *where*

$$f'(c) = \frac{\psi'(\tanh(c))\operatorname{sech}^2(c)}{1-(\psi(\tanh(c)))^2} = 0.8807\ldots = a.$$

Proof. Differentiating f with respect to x, we get

$$f'(x) = \frac{\psi'(\tanh(x))\operatorname{sech}^2(x)}{1-(\psi(\tanh(x)))^2} = \frac{F(x)}{G(x)}.$$

We see that f' is positive and decreasing, because

$$\begin{aligned}
F'(x) &= \frac{\psi''(\tanh(x))}{\cosh(x)^4} - 2\frac{\psi'(\tanh(x))\tanh(x)}{\cosh(x)^2} \\
&< \frac{1}{\cosh(x)^4}\left(\frac{1}{\tanh(x)} - 2\psi'(\tanh(x))\right) - 2\frac{\psi'(x)\tanh(x)}{\cosh(x)^2} \\
&< \frac{1}{\cosh(x)^4}\left(\frac{1}{\tanh(x)} - 2\left(\frac{1}{\tanh(x)} + \frac{1}{2\tanh(x)^2}\right)\right) - 2\frac{\psi'(x)\tanh(x)}{\cosh(x)^2}
\end{aligned}$$

$$= -\frac{1}{\cosh(x)^4}\left(\frac{1}{\tanh(x)} + \frac{1}{\tanh(x)^2}\right) - 2\frac{\psi'(x)\tanh(x)}{\cosh(x)^2} < 0,$$

by Lemma 1 2) & 3). Clearly, $G(x)$ is increasing, hence f is concave. The concavity of the function implies

$$f\left(\frac{x+y}{2}\right) > \frac{f(x)+f(y)}{2},$$

which gives

$$\psi\left(\tanh\left(\frac{x+y}{2}\right)\right) > \tanh\left(\frac{\operatorname{artanh}(\psi(\tanh(x))) + \operatorname{artanh}(\psi(\tanh(y)))}{2}\right).$$

Write $r = \tanh(x), s = \tanh(y), R = \psi(r), S = \psi(s)$, and

$$u = \operatorname{artanh}(\psi(\tanh(x))), \quad v = \operatorname{artanh}(\psi(\tanh(y))).$$

We get 1) by using

$$\tanh\left(\frac{x+y}{2}\right) = \frac{\tanh(x+y)}{1+\sqrt{1-\tanh^2(x+y)}} = \frac{r+s}{1+rs+r's'},$$

$$\tanh\left(\frac{u+v}{2}\right) = \frac{R+S}{1+RS+R'S'}.$$

The derivative $f'(x)$ tends to a when x tends to c. By the Mean Value Theorem, we get $f(r) - f(s) < a(r-s)$. This is equivalent to

$$\frac{1}{2}\log\left(\frac{1+\psi(\tanh(r))}{1-\psi(\tanh(r))}\right) - \frac{1}{2}\log\left(\frac{1+\psi(\tanh(s))}{1-\psi(\tanh(s))}\right) < a(r-s),$$

hence 2) follows, and this completes the proof. □

Lemma 4. *[18, Thm 1.7] Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function, and for $c_2 \neq 0$ define*

$$g(x) = \frac{f(x^{c_2})}{(f(x))^{c_2}}.$$

We have the following:

1) *if $h(x) = \log(f(e^x))$ is a convex function, then $g(x)$ is monotone increasing for $c_2, x \in (0,1)$ or $c_2, x \in (1,\infty)$ or $c < 0, x > 1$, and monotone decreasing for $c_2 \in (0,1), x > 1$ or $c_2 > 1$, $x \in (0,1)$ or $c_2 < 0$, $x \in (0,1)$,*

2) *if $h(x)$ is a concave function, then $g(x)$ is monotone increasing for $c_2 \in (0,1)$, $x > 1$ or $c_2 > 1$, $x \in (0,1)$ or $c_2 < 0$, $x \in (0,1)$ and monotone decreasing for c_2, $x \in (0,1)$ or $c_2 > 1$, $x > 1$ or $c_2 < 0$, $x > 1$.*

Lemma 5. *[11, Lemma 2.1] Let us consider the function $f\colon (a,\infty) \to \mathbb{R}$, where $a \geqslant 0$. If the function g, defined by*

$$g(x) = \frac{f(x)-1}{x},$$

is increasing on (a,∞), then for the function h, defined by $h(x) = f(x^2)$, we have the following Grünbaum-type inequality:

$$1 + h(z) \geqslant h(x) + h(y), \tag{5}$$

where $x, y \geqslant a$ and $z^2 = x^2 + y^2$. If the function g is decreasing, then inequality (5) is reversed.

Theorem 4. *The following inequalities hold for $r, s \in (c,\infty)$:*

1) $\psi(\sqrt{rs}) \geqslant \sqrt{\psi(r)\psi(s)}$, *equality holds with $r = s$,*
2) $\psi(r^k) < \psi(r)^k, \quad k \in (0,1)$,
3) $\psi(r)^k < \psi(r^k), \quad k > 1$,
4) $\dfrac{r + s + \psi(r+s)}{r\,\psi(s) + s\,\psi(r)} \geqslant \dfrac{r+s}{rs}, \quad r,\ s > c^2.$

Proof. Let $f(x) = \log(\psi(e^x))$, $x > t = 0.379554$, where t is the solution of the equation $e^t = c$. Differentiating f with respect to x, we get, by Lemma 1 5) & 2),

$$\begin{aligned}
f''(x) &= \frac{e^x\left(\psi\left(e^x\right)\left(\psi'\left(e^x\right)+e^x\psi''\left(e^x\right)\right)-e^x\psi'\left(e^x\right)^2\right)}{\psi\left(e^x\right)^2} \\
&< \frac{e^x\left(\psi\left(e^x\right)\left(\psi'\left(e^x\right)+1/e^x-2\psi'(e^x)\right)-e^x\psi'\left(e^x\right)^2\right)}{\psi\left(e^x\right)^2} \\
&< \frac{e^x}{\psi\left(e^x\right)^2}\left(\psi\left(e^x\right)\left(\psi'\left(e^x\right)+\frac{1}{e^x}-2\psi'(e^x)\right)-e^x\left(\frac{1}{e^x}+\frac{1}{2e^{2x}}\right)^2\right) \\
&< \frac{e^x}{\psi\left(e^x\right)^2}\left(\psi\left(e^x\right)\left(\frac{1}{e^x}+\frac{1}{e^{2x}}+\frac{1}{e^x}-2\left(\frac{1}{e^x}+\frac{1}{2e^{2x}}\right)\right)\right) \\
&\quad -\frac{e^{2x}}{\psi\left(e^x\right)^2}\left(\frac{1}{e^x}+\frac{1}{2e^{2x}}\right)^2 = -\frac{e^{2x}}{\psi\left(e^x\right)^2}\left(\frac{1}{e^x}+\frac{1}{2e^{2x}}\right)^2 < 0.
\end{aligned}$$

Hence, f is concave; this implies

$$\frac{\log(\psi(e^x)) + \log(\psi(e^y))}{2} \leqslant \log(\psi(e^{(x+y)/2})).$$

If we let $r = e^x$ and $s = e^y$, we get (1).

For 2), since $\log(\psi(e^x))$ is concave by part 1), then, by Lemma 4 2), the function $\psi(x^k)/\psi(x)^k$ is monotone increasing for $k \in (0,1)$. This implies

$$\frac{\psi(x^0)}{\psi(x)^0} < \frac{\psi(x^k)}{\psi(x)^k} < \frac{\psi(x^1)}{\psi(x)^1},$$

and part (2) follows. The proof of part 3) is similar.

For the proof of part 4), let

$$f_2(x) = \frac{\psi(x)/x - 1}{x}, \quad x > c.$$

Differentiating f_1 with respect x and using Lemma 1 1) & 2) and the inequality $\log(1+x) \leqslant x(2+x)/(2(1+x)), x \geqslant 0$, [27], we get

$$f_2'(x) = \frac{\psi'(x)x + x - 2\psi(x)}{x^3} > \frac{2x^2 + 2x - 4x\log(x) + 3}{2x^4}$$
$$\geqslant \frac{2x^2 + 2x - 2(x^2-1) + 3}{2x^4} = \frac{2x+5}{2x^4} > 0.$$

Now the proof of part 4) follows from Lemma 5. □

Acknowledgment. The authors sincerely thank the anonymous referees for their valuable comments and constructive suggestions, which have significantly improved the quality and clarity of this paper.

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Received December 10, 2024.
In revised form, May 11, 2025.
Accepted May 11, 2025.
Published online May 21, 2025.

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