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$\mathcal{I}^{\mathcal{K}}$-SEQUENTIAL AND $\mathcal{I}^{\mathcal{K}}$-FRÉCHET-URYSOHN SPACES

Abstract. Notions of $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and $\mathcal{I}^{\mathcal{K}}$-sequential spaces are studied by letting ideals $\mathcal{I}$, $\mathcal{K}$ of subsets of natural numbers to play measurable role in the well-established concepts of Fréchet-Urysohn and sequential spaces. Relation among those spaces in new and old setting have been established by introducing $\mathcal{I}^{\mathcal{K}}$-quotient maps and $\mathcal{I}^{\mathcal{K}}$-covering maps.

Key words: *$\mathcal{I}^{\mathcal{K}}$-quotient map, $\mathcal{I}^{\mathcal{K}}$-covering map, $\mathcal{I}^{\mathcal{K}}$-sequential space, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space*

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1. Introduction. In 1973, J. R. Boone and F. Siwiec [1] introduced the concept of sequentially quotient maps, which are the convergent sequence analogs of the bi-quotient maps of Michael [18]. The notions of sequential spaces and sequentially open subsets of a space were introduced by Franklin [8]. In [17], the notions of statistically Fréchet-Urysohn and statistically sequential spaces have been defined and studied in detail in [25]. Statistical convergence introduced by H. Fast [6] is an extension of the concept of convergence of sequence of real numbers. During last four decades, many mathematicians explored and generalized that concepts in various directions ([3], [13], [17], [22], etc.). Two interesting generalizations of statistical convergence are $\mathcal{I}$ and $\mathcal{I}^*$-convergence [12]. After a long time, in the year 2011, M. Macaj and M. Sleziak introduced the concept of $\mathcal{I}^{\mathcal{K}}$-convergence, as a generalization of $\mathcal{I}^*$-convergence. In 2022, C. Choudhury and S. Debnath [2] defined the notions of $\mathcal{I}^{\mathcal{K}}$-supremum, $\mathcal{I}^{\mathcal{K}}$-infimum, $\mathcal{I}^{\mathcal{K}}$-limit superior and $\mathcal{I}^{\mathcal{K}}$-limit inferior and studied their relations. Recently, the concept of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers was introduced and explored its properties [11]. Several properties of $\mathcal{I}^{\mathcal{K}}$-convergence of functions have been studied in [4], [20], [21].

Here are some basic definitions and findings provided as a ready references that will be used in the sequel.

An ideal $\mathcal{I}$ on an arbitrary set X is a family $\mathcal{I} \subset 2^X$ that is closed under finite unions and taking subsets [14]. An ideal $\mathcal{I}$ is called trivial if $\mathcal{I} = \varnothing$ or X in $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if it contains all the singleton sets [14]. The class of all finite subsets of $\mathbb{N}$ is an admissible ideal on $\mathbb{N}$, denoted by Fin.

Various examples of non-trivial admissible ideals are given in [12]. Suppose $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$. A sequence (x_n) in a topological space X is said to be $\mathcal{I}$-convergent to l in X if for any open set U containing l, $\{n \in \mathbb{N}\colon x_n \notin U\} \in \mathcal{I}$ [15]. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}^*$-convergent to $l \in X$ if there exists a set $M \in \mathcal{F}(\mathcal{I})$, such that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_n = x_n$, $n \in M$, and $y_n = l$, $n \in \mathbb{N}\backslash M$ is Fin-convergent to l. In addition, $\mathcal{I}^{\mathcal{K}}$-convergence is defined by replacing Fin by an arbitrary ideal $\mathcal{K}$ on $\mathbb{N}$.

Let us recall the notion of sequential spaces. A subset C of a topological space X is called sequentially closed if no sequence in C converges to a point in $X\backslash C$. A topological space X is said to be sequential if each sequentially closed subset of X is closed [8]. Every first countable space is a sequential space. Suppose X, Y are topological spaces and $f : X \to Y$ is an onto map; f is called a quotient map provided a subset U is open in Y if and only if $f^{-1}(U)$ is open in X, and f is called a sequentially quotient map provided a subset U is sequentially open in Y if and only if $f^{-1}(U)$ is sequentially open in X [1]. f is said to be sequence-covering if whenever (y_i) is a sequence in Y converging to some point l in Y, there exists a sequence (x_i) of points $x_i \in f^{-1}(y_i)$ for all $i \in \mathbb{N}$ and $p \in f^{-1}(l)$, such that (x_n) converges to p [1]. Every sequence-covering mapping is sequentially quotient. A topological space X is said to be Fréchet-Urysohn if for each subset C of X and $x \in \bar{C}$, there exists a sequence in C converging to x [8]. Every Fréchet-Urysohn space is sequential, but the reverse implication may not hold [8].

Before entering into the main discussion, let us take a look at some of the ones that will be followed throughout the article:

- A sequence is a mapping whose domain is a cofinal subset of $\mathbb{N}$. Let $x = (x_n)_{n\in L}$ be a sequence in a topological space X and M be a cofinal subset of L. Then call $(x_n)_{n\in M}$ a subsequence of $x = (x_n)_{n\in L}$.
- Nonthin subsets of natural numbers were introduced by J. A. Fridy [9] in terms of natural density [10]. Inspired by the notion of nonthin sets, $\mathcal{I}$-nonthin subsets of natural numbers are defined in [23]. A sequence $(x_n)_{n\in A}$ in X is said to be $\mathcal{I}$-thin if $A \in \mathcal{I}$, otherwise it is

called $\mathcal{I}$-nonthin, where $A \subset \mathbb{N}$ and $\mathcal{I}$ is an ideal on $\mathbb{N}$ [23].

- For $M \subset \mathbb{N}$, $\mathcal{I}|_M = \{A \cap M; A \in \mathcal{I}\}$ is an ideal on M [16]. $\mathcal{I}|_M$ is nontrivial if $M \notin \mathcal{I}$.
- $\mathcal{I}$, $\mathcal{K}$ stand for nontrivial admissible ideal on $\mathbb{N}$, unless otherwise stated.
- all mappings are onto.

2. Main Results. In this section, the notion of $\mathcal{I}^{\mathcal{K}}$-sequential space is introduced, and we show that $\mathcal{I}^{\mathcal{K}}$-sequential space may not be sequential.

Definition 1. *The $\mathcal{I}^{\mathcal{K}}$-closure of a subset C of a topological space X is denoted by $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = \{x \in X\colon$ there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in A}$ in X, such that $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to $x\}$.*

Theorem 1. *Let $\mathcal{K} \subset \mathcal{I}$. For any subset C of a topological space X, $C \subset \overline{C}^{\mathcal{I}^{\mathcal{K}}} \subset \bar{C}$, where $\bar{C}$ is the closure of C. Furthermore, if X is first countable, $\bar{C} = \overline{C}^{\mathcal{I}^{\mathcal{K}}}$.*

Proof. Let $x \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in A}$ in C, which $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to x. Therefore, there exists $M \in \mathcal{F}(\mathcal{I}|_A)$, such that the sequence $(y_n)_{n\in A}$ given by $y_n = x_n$ if $n \in M$ and $y_n = x$ if $n \in A\backslash M$ is $\mathcal{K}$-convergent to x. For any open set U containing x, $\{n \in A\colon y_n \in U\} \in \mathcal{F}(\mathcal{K}|_A)$. Since $\mathcal{K} \subset \mathcal{I}$, the set $\{n \in A\colon y_n \in U\} \in \mathcal{F}(\mathcal{I}|_A)$ and, so, $\{n \in A\colon x_n \in U\} \in \mathcal{F}(\mathcal{I}|_A)$. Therefore, there is $p \in A$, such that $p \in \{n \in A\colon x_n \in U\}$. Then $x_p \in C \cap U$ and, hence, $x \in \bar{C}$. Suppose X is first countable and $x \in \bar{C}$. Then there exists a sequence (x_n) in C, such that (x_n) is convergent to x. Since $\mathcal{I}$ and $\mathcal{K}$ are admissible ideals on $\mathbb{N}$, (x_n) is $\mathcal{K}$-convergent and, so, (x_n) $\mathcal{I}^{\mathcal{K}}$-converges to x. Thus, $x \in \overline{C}^{\mathcal{I}^{\mathcal{K}}}$. □

Definition 2. *A subset C of a topological space X is called $\mathcal{I}^{\mathcal{K}}$-closed if $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = C$.*

Theorem 2. *For any subset H of a topological space X, the following are equivalent:*

(a) H is $\mathcal{I}^{\mathcal{K}}$-open.

(b) for any $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in X with $(x_n)_{n\in L}$, $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$, $\{n \in L\colon x_n \in H\} \notin \mathcal{K}$.

(c) for any $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in X with $(x_n)_{n\in L}i$, $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$, $|\{n \in L\colon x_n \in H\}| = \omega$.

Proof. $(a) \implies (b)$ Suppose H is $\mathcal{I}^{\mathcal{K}}$-open and $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in X, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in H$. If possible, let $M = \{n \in L : x_n \in H\} \in \mathcal{K}$. Then $M \neq L$ and, so, $X \neq H$. Let $p \in X\backslash H$. Define a sequence $(y_n)_{n\in L}$ in X given by $y_n = p$, $n \in M$, and $y_n = x_n$, $n \notin L\backslash M$. Clearly, $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. Since $X\backslash H$ is $\mathcal{I}^{\mathcal{K}}$-closed and $(y_n)_{n\in L}$ is a sequence in $X\backslash H$, $x \in X\backslash H$, which leads to a contradiction. Hence, $\{n \in L : x_n \in H\} \notin \mathcal{K}$.
$(b) \implies (c)$ It is obvious, as the ideal $\mathcal{I}$ is an admissible ideal on $\mathbb{N}$.
$(c) \implies (a)$ Suppose $X\backslash H$ is not $\mathcal{I}^{\mathcal{K}}$-closed in X. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $X\backslash H$, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to some point $x \in H$. So, $\{n \in L : x_n \in H\}$ is an empty set, which leads to a contradiction. Hence, H is $\mathcal{I}^{\mathcal{K}}$-open in X. □

Definition 3. *A topological space is said to be $\mathcal{I}^{\mathcal{K}}$-sequential if every $\mathcal{I}^{\mathcal{K}}$-closed set is closed.*

Suppose C is an $\mathcal{I}^{\mathcal{K}}$-closed subset of a topological space X. If (x_n) is a sequence in C, such that (x_n) converges to $x \in X$, then (x_n) $\mathcal{K}|_{\mathbb{N}}$-converges to x. Since C is $\mathcal{I}^{\mathcal{K}}$-closed, $x \in C$. Thus, C is sequentially closed. If X is a sequential space, then C is closed. So, every sequential space is $\mathcal{I}^{\mathcal{K}}$-sequential. But the converse may not be true.

Theorem 3. *Every sequential space is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Example 1. Suppose $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $\mathcal{K}$ is a maximal ideal on $\mathbb{N}$. Consider the space $\sum(\mathcal{K})$ defined in [26, Example 2.7] as follows:
Take the set $Y = \mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$. A topology on Y consists of each $\{n\}$ and sets G containing ∞ of the form $G = \{\infty\} \cup (\mathbb{N}\backslash A)$, where $A \in \mathcal{K}$. Denote the set Y equipped with this topology by $\sum(\mathcal{K})$. Suppose G is an $\mathcal{I}^{\mathcal{K}}$-open subset of $\sum(\mathcal{K})$. Let us assume that $\infty \in G$. Consider the sequence (n), which $\mathcal{I}^{\mathcal{K}}$-converges to ∞ in $\sum(\mathcal{K})$. By Theorem 2, it follows that $\{n \in \mathbb{N} : n \in G\} \notin \mathcal{K}$. Therefore, $G\backslash\{\infty\} \notin \mathcal{K}$. Since $\mathcal{K}$ is a maximal ideal of $\mathbb{N}$, $\mathbb{N}\backslash G \in \mathcal{K}$. Therefore, $G = \{\infty\} \cup (\mathbb{N}\backslash(\mathbb{N}\backslash G))$ is open in $\sum(\mathcal{K})$. Hence, $\sum(\mathcal{K})$ is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Moreover, $\sum(\mathcal{K})$ is a Hausdorff space, but not a k-space [26, Example 2.9]. Again, since every sequential space is a k-space [19], $\sum(\mathcal{K})$ is not a sequential space.

Definition 4. *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-continuous if for every $\mathcal{I}$-nonthin sequence $(x_i)_{i\in P}$ in Y, which is $(\mathcal{I}|_P)^{\mathcal{K}}$-convergent to x, $(f(x_i))_{i\in P}$ $(\mathcal{I}|_P)^{\mathcal{K}}$-converges to $f(x)$.*

Theorem 4. *Let X and Y be topological spaces. A function $f : X \to Y$*

is $\mathcal{I}^{\mathcal{K}}$-continuous if and only if $f^{-1}(B)$ is $\mathcal{I}^{\mathcal{K}}$-closed for every $\mathcal{I}^{\mathcal{K}}$-closed subset B of Y.

Proof. Suppose f is an $\mathcal{I}^{\mathcal{K}}$-continuous function and B is an $\mathcal{I}^{\mathcal{K}}$-closed subset of Y. Let $x \in \overline{f^{-1}(B)}^{\mathcal{I}^{\mathcal{K}}}$. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $f^{-1}(B)$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in X$. So $(f(x_n))_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $f(x)$. Since B is $\mathcal{I}^{\mathcal{K}}$-closed, $x \in f^{-1}(B)$.
Conversely, for every $\mathcal{I}^{\mathcal{K}}$-closed subset B of Y, $f^{-1}(B)$ is an $\mathcal{I}^{\mathcal{K}}$-closed subset of X. Suppose f is not $\mathcal{I}^{\mathcal{K}}$-continuous. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in M}$ in X, which $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to $x \in X$, but $(f(x_n))_{n\in M}$ does not $(\mathcal{I}|_M)^{\mathcal{K}}$-converge to $f(x)$. For all $T \in \mathcal{F}(\mathcal{I}|_M)$, such that the sequence $(y_n)_{n\in M}$ given by $y_n = f(x_n)$, $n \in T$ and $y_n = f(x)$, $n \in M\backslash T$ does not $\mathcal{K}$-converge to $f(x)$. Therefore, there exists an open set U containing $f(x)$, such that $\{n \in M : y_n \notin U\} \notin \mathcal{K}|_M$. As $\{n \in M : f(x_n) \notin U\} \supset \{n \in M : y_n \notin U\}$, $P = \{n \in M : f(x_n) \notin U\} \notin \mathcal{K}|_M$. Again, $Y\backslash U$ is $\mathcal{I}^{\mathcal{K}}$-closed in Y, because $Y\backslash U$ is closed in Y. So $f^{-1}(Y\backslash U)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Since $(x_n)_{n\in P}$ $(\mathcal{I}|_P)^{\mathcal{K}}$-converges to x, $x \in \overline{f^{-1}(Y\backslash U)}^{\mathcal{I}^{\mathcal{K}}} = f^{-1}(Y\backslash U)$. Therefore, $f(x) \in Y\backslash U$, which is a contradiction. $\square$

Corollary 1. *Suppose Y and Z are topological spaces. The following are equivalent for a function $\phi : Y \to Z$:*

(a) *ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous.*

(b) *$\phi^{-1}(F)$ is $\mathcal{I}^{\mathcal{K}}$-closed for every $\mathcal{I}^{\mathcal{K}}$-closed subset F of Z.*

(c) *$\phi^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-open for every $\mathcal{I}^{\mathcal{K}}$-open subset G of Z.*

3. $\mathcal{I}^{\mathcal{K}}$-quotient map and $\mathcal{I}^{\mathcal{K}}$-covering map. In this section, the notion of $\mathcal{I}^{\mathcal{K}}$-quotient map is introduced, which is an extension of $\mathcal{I}^{\mathcal{K}}$-continuous map. Also, the concept of $\mathcal{I}^{\mathcal{K}}$-covering map is defined and relation between $\mathcal{I}^{\mathcal{K}}$-quotient map and $\mathcal{I}^{\mathcal{K}}$-covering map are studied. Suppose $(x_n)_{n\in L}$ is any $\mathcal{I}$-nonthin sequence in a topological space X. $(x_n)_{n\in L}$ is said to be $\mathcal{I}$-eventually constant at x if $\{n \in L : x_n \neq x\} \in \mathcal{I}|_L$ [24]. Every eventually constant sequence is $\mathcal{I}$-eventually constant. But the reverse implication may not hold [24].

Definition 5. *A function $f : X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-presequential if for any $\mathcal{I}$-nonthin sequence $(y_n)_{n\in M}$ in Y with $(y_n)_{n\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to y and $(y_n)_{n\in M}$ non $\mathcal{I}$-eventually constant at y, $\cup\{f^{-1}(y_n) : n \in M, y_n \neq y\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed.*

Definition 6. *A mapping $\phi\colon X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-quotient provided that a set G is $\mathcal{I}^{\mathcal{K}}$-closed* ($\mathcal{I}^{\mathcal{K}}$-open) in Y if and only if $\phi^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-closed (resp. $\mathcal{I}^{\mathcal{K}}$-open) in X.

Theorem 5. *Suppose $\mathcal{I}$, $\mathcal{K}$ are ideals on $\mathbb{N}$ and $\phi\colon X \to Y$ is $\mathcal{I}^{\mathcal{K}}$-continuous function. Then the following are equivalent:*

(a) *ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential.*

(b) *For each non $\mathcal{I}^{\mathcal{K}}$-closed subset C of Y, $\phi^{-1}(C)$ is non $\mathcal{I}^{\mathcal{K}}$-closed subset of X.*

(c) *For each non $\mathcal{I}^{\mathcal{K}}$-open subset G of Y, $\phi^{-1}(G)$ is non $\mathcal{I}^{\mathcal{K}}$-open subset of X.*

Proof. The condition (b) and (c) are equivalent by considering complement.

For (b) $\implies$ (a), let $\alpha = (\alpha_i)_{i\in M}$ be any $\mathcal{I}$-nonthin sequence in Y, such that $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to ξ and is non $\mathcal{I}$-eventually constant at ξ. If $L = \{i \in M\colon \alpha_i \neq \xi\}$, ξ is not equal to any $(\alpha_i)_{i\in L}$. Again, $\cup\{\phi^{-1}(\alpha_i)\colon i \in M$ and $\alpha_i \neq \xi\} = \phi^{-1}(Im\ \alpha\backslash\{\xi\})$. Since $Im\ \alpha\backslash\{\xi\}$ of Y is not $\mathcal{I}^{\mathcal{K}}$-closed, $\cup\{\phi^{-1}(\alpha_i)\colon i \in M$ and $\alpha_i \neq \xi\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed.

(a) $\implies$ (b) Suppose ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential and $\mathcal{I}^{\mathcal{K}}$-continuous. Let C be a non $\mathcal{I}^{\mathcal{K}}$-closed subset of Y. Then there exists an $\mathcal{I}$-nonthin sequence $\alpha = (\alpha_i)_{i\in M}$ in C, which is $(\mathcal{I}|_M)^{\mathcal{K}}$ converging to some point ξ in $Y\backslash C$. Therefore, ξ is not equal to any α_i. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential, then the set $G = \phi^{-1}(Im\ \alpha)$ is not $\mathcal{I}^{\mathcal{K}}$-closed. Thus there exists a sequence $(\gamma_i)_{i\in L}$ in G with $L \subset M$, such that $\phi(\gamma_i) = \alpha_i$ for all $i \in L$. So, $(\gamma_i)_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to a point η in $X\backslash G$. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous, so the sequence $(\alpha_i)_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $\phi(\eta) = \xi$, $\eta \notin \phi^{-1}(C)$. As $(\alpha_i)_{i\in L}$ is in G, $(\alpha_i)_{i\in L}$ is in $\phi^{-1}(C)$. Therefore, $\phi^{-1}(C)$ is not an $\mathcal{I}^{\mathcal{K}}$-closed subset in X. $\square$

Corollary 2. *A mapping is $\mathcal{I}^{\mathcal{K}}$-quotient if and only if it is both $\mathcal{I}^{\mathcal{K}}$-continuous and $\mathcal{I}^{\mathcal{K}}$-presequential.*

Definition 7. *A mapping $\phi\colon X \to Y$ is said to be $\mathcal{I}^{\mathcal{K}}$-covering if for every $\mathcal{I}$-nonthin sequence $(\beta_i)_{i\in M}$ in Y that $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to β in Y, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in M}$ with $\alpha_i \in \phi^{-1}(\beta_i)$, for $i \in M$ and $\alpha \in \phi^{-1}(\beta)$, such that $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to α.*

Suppose $\phi\colon X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-covering mapping. If G is a non $\mathcal{I}^{\mathcal{K}}$-closed subset of Y, then there exists an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in Y, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to some point say y, $y \notin G$. As ϕ is

$\mathcal{I}^{\mathcal{K}}$-covering, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ of points $x_n \in \phi^{-1}(y_n)$ for all $n \in L$ and $x \in \phi^{-1}(y)$, such that $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. But $x \notin \phi^{-1}(G)$. Therefore, $\phi^{-1}(G)$ is not $\mathcal{I}^{\mathcal{K}}$-closed.

So if ϕ is an $\mathcal{I}^{\mathcal{K}}$-continuous $\mathcal{I}^{\mathcal{K}}$-covering mapping, ϕ satisfies condition (b) of Theorem 5 and, so, ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential. Therefore, ϕ is $\mathcal{I}^{\mathcal{K}}$-quotient.

Theorem 6. *Every $\mathcal{I}^{\mathcal{K}}$-continuous $\mathcal{I}^{\mathcal{K}}$-covering mapping is $\mathcal{I}^{\mathcal{K}}$-quotient.*

Theorem 7. *A one-to-one $\mathcal{I}^{\mathcal{K}}$-quotient mapping is $\mathcal{I}^{\mathcal{K}}$-covering.*

Proof. Suppose $\phi\colon X \to Y$ is an one-to-one $\mathcal{I}^{\mathcal{K}}$-quotient mapping. Let $(\beta_i)_{i\in M}$ be an $\mathcal{I}$-nonthin sequence in Y, which $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to a point $\beta \in Y$. Without loss of generality, let us assume that $(\beta_i)_{i\in M}$ consists of distinct points. Let $\alpha_i = \phi^{-1}(\beta_i)$ and $\alpha = \phi^{-1}(\beta)$. If possible, let $(\alpha_i)_{i\in M}$ be not $(\mathcal{I}|_M)^{\mathcal{K}}$-convergent to α. For any set $P \in \mathcal{F}(\mathcal{I}|_M)$, consider a sequence $(\gamma_i)_{i\in M}$ given by $\gamma_i = \alpha_i$, $i \in P$, and $\gamma_i = \alpha$, $i \in M\backslash P$ is not $\mathcal{K}|_M$-convergent to α. Then there exists an open set W containing α, such that the set $L = \{i \in M\colon \gamma_i \notin W\} \notin \mathcal{K}|_M$. Thus the sequence $(\gamma_i)_{i\in L}$ is not in W. For each $i \in L\backslash P$, $\phi(\gamma_i) = \beta_i$, which shows that $(\phi(\gamma_i))_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Since ϕ is $\mathcal{I}^{\mathcal{K}}$-quotient, ϕ is $\mathcal{I}^{\mathcal{K}}$-presequential. Then $\cup\{\phi^{-1}(\beta_i)\colon i \in K \text{ and } \beta_i \neq \beta\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. So, there exists an $\mathcal{I}$-nonthin sequence $(z_i)_{i\in T}$ in $\cup\{\phi^{-1}(\beta_i)\colon i \in K \text{ and } \beta_i \neq \beta\}$, which $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to some point l in X. There is a set $A \in \mathcal{F}(\mathcal{I}|_T)$, such that the sequence $(u_i)_{i\in T}$ is given by $u_i = z_i$, $i \in A$ and $u_i = l$, $i \in T\backslash A$ $\mathcal{K}|_T$-converges to l. Again, since ϕ is $\mathcal{I}^{\mathcal{K}}$-continuous, then $(\phi(u_i))_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\phi(l)$. For each $i \in A$, $\phi(u_i) = \beta_i$, and $(\beta_i)_{i\in A}$, $(\mathcal{I}|_A)^{\mathcal{K}}$-converges to β. Therefore, $\phi(l) = \beta$ and $l = \phi^{-1}(\beta) = \alpha$. As W is an open set containing $\alpha = l$, $\{i \in T\colon u_i \in W\} \notin \mathcal{K}|_T$. So, $\{i \in L\colon \alpha_i \in W\} \notin \mathcal{K}|_L$, which leads to a contradiction. Hence, $(\alpha_i)_{i\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to α. □

Corollary 3. *A one-to-one $\mathcal{I}^{\mathcal{K}}$-continuous mapping is $\mathcal{I}^{\mathcal{K}}$-quotient if and only if the mapping is $\mathcal{I}^{\mathcal{K}}$-covering.*

Theorem 8. *For an $\mathcal{I}^{\mathcal{K}}$-continuous mapping $h\colon X \to Y$, the following are equivalent:*

(a) h is an $\mathcal{I}^{\mathcal{K}}$-quotient map.

(b) for each $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in Y, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β say, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in T}$ with $\alpha_{m_i} \in h^{-1}(y_{n_i})$, such that $(\alpha_i)_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$,

where $T = \{m_1 < m_2 < \ldots\}$ and $\{n_1 < n_2 < \ldots\}$ are $\mathcal{I}$-nonthin subsets of L.

(c) *for each β in the $\mathcal{I}^{\mathcal{K}}$-closure of a subset D of Y, there exists a point $\alpha \in h^{-1}(\beta)$, such that α is in the $\mathcal{I}^{\mathcal{K}}$-closure of $h^{-1}(D)$.*

Proof. $(a) \implies (b)$ Suppose $(y_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in Y, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Without loss of generality, let $y_n \neq \beta$ for each $n \in L$. So, $\{y_n : n \in L\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. As h is $\mathcal{I}^{\mathcal{K}}$-presequential, $\cup\{h^{-1}(y_n) : n \in L\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed. Again, since h is $\mathcal{I}^{\mathcal{K}}$-continuous, $\cup\{h^{-1}(y_n) : n \in L\} \cup h^{-1}(\beta)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Therefore, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_n)_{n\in M}$ in $\cup\{h^{-1}(y_n) : n \in L\}$ that $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to some point $\alpha \in h^{-1}(\beta)$. For each $n \in L$, $h^{-1}(y_n)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Therefore, for each $n \in L$ there is at most an $\mathcal{I}$-thin subsequence $(\alpha_n)_{n\in M_1}$ of $(\alpha_n)_{n\in M}$, which belong to $h^{-1}(y_n)$. So, there exists an $\mathcal{I}$-nonthin set $P = \{n_1 < n_2 < \ldots < n_k < \ldots\} \subset L$, such that $T = \{i \in M : \alpha_i \in h^{-1}(y_{n_i})\} \notin \mathcal{I}$. Thus, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in T}$ with $\alpha_{m_i} \in h^{-1}(y_{n_i})$, $T = \{m_1 < m_2 < \ldots\}$, such that $(\alpha_i)_{i\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$.

$(b) \implies (c)$ Let $\beta \in \overline{D}^{\mathcal{I}^{\mathcal{K}}}$. Without loss of generality, let $\beta \notin D$. There exists an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in D, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to β. Then there is an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in M}$ in X, such that $x_{m_k} \in h^{-1}(y_{n_k})$, where $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ and $\{n_1 < n_2 < \ldots < n_k < \ldots\} \subset L$ and $(x_n)_{n\in M}$ $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to $\alpha \in h^{-1}(\beta)$. Since $x_n \in h^{-1}(D)$, for each $n \in M$, so α is in the $\mathcal{I}^{\mathcal{K}}$-closure of $h^{-1}(D)$.

$(c) \implies (a)$ Suppose h is not an $\mathcal{I}^{\mathcal{K}}$-presequential mapping. Then there exists a non $\mathcal{I}^{\mathcal{K}}$-closed subset D of Y, such that $h^{-1}(D)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Suppose β is a point in the $\mathcal{I}^{\mathcal{K}}$-closure of D and $\beta \notin D$. Then $h^{-1}(\beta) \notin h^{-1}(D)$. Since $h^{-1}(D)$ is $\mathcal{I}^{\mathcal{K}}$-closed, there does not exist a point $\alpha \in h^{-1}(\beta)$, such that $\alpha \in \overline{h^{-1}(D)}^{\mathcal{I}^{\mathcal{K}}}$. Hence, h is $\mathcal{I}^{\mathcal{K}}$-presequential. $\square$

Example 2. Let $\mathcal{I} = \mathcal{K} = \mathcal{P}(2\mathbb{N}) \cup Fin$, $\mathcal{P}(2\mathbb{N})$ be the power set of $2\mathbb{N}$, and Fin be the class of all finite subsets of $\mathbb{N}$. Consider $I = [0,1]$ with the usual topology and for each $\alpha \in I$, $S_\alpha = \{x_{\alpha,n} : n \in \mathbb{N}\}$ and $S'_\alpha = S_\alpha \cup \{x_\alpha\}$. A topology τ on S'_α consists of each $\{x_{\alpha,n}\}$ and sets U containing x_α equals to $\{x_{\alpha,n} : n \geqslant n_0\} \cup \{x_\alpha\}$, for some $n_0 \in \mathbb{N}$. Suppose X is a topological sum of a collection $\{I, S'_\alpha : \alpha \in I\}$. Let $Y = (\oplus S_\alpha) \oplus I$ be the space with a topology τ_1 that consists of each $\{x_{\alpha,n}\}$ and sets U containing α of the form $\{x_{\alpha,n} : n \geqslant m\} \cup G$, where G is an open set containing α in I and

$m \in \mathbb{N}$. Consider the map $f\colon X \to Y$ defined by $f(x) = x$, if $x = x_{\alpha,n} \in S_\alpha$ and $f(x) = \alpha$, if $x = x_\alpha$ or $x \in I$.

Suppose $S = (y_n)_{n\in M}$ is an $\mathcal{I}$-nonthin sequence in Y that $(\mathcal{I}|_M)^{\mathcal{K}}$-converges to y. So $y \in I$. Let $S_1 = S \cap S_y$ and $S_2 = S \cap I$. Since S is an $\mathcal{I}$-nonthin sequence, either S_1 or S_2 must be $\mathcal{I}$-nonthin. Again, S_1 and S_2 are $\mathcal{I}^{\mathcal{K}}$-convergent in X with its image being an $\mathcal{I}$-nonthin subsequence of S. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map. Now, suppose (p_n) is a sequence in I converging to α in I. A sequence $S = (z_n)$ in Y is defined by $z_n = x_{\alpha,n}$, if $n \in 4\mathbb{N}+1$ and $z_n = p_n$, if $n \notin 4\mathbb{N}+1$. Therefore, (z_n) converges to α in Y, so (z_n) $\mathcal{I}^{\mathcal{K}}$-converges to α. Let $S_1 = S \cap S_\alpha$ and $S_2 = S \cap I$. Then S_1 and S_2 are $\mathcal{I}$-nonthin sequences in X. Thus S_1 $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x_α and S_2 $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to α, where $L = 4\mathbb{N}+1$, $T = \mathbb{N}\backslash 4\mathbb{N}+1$. Since X is Hausdorff, so, corresponding to S, there is no $\mathcal{I}$-nonthin sequence in X, whose image is S. Hence, f is not an $\mathcal{I}^{\mathcal{K}}$-covering map.

Theorem 9. *$\mathcal{I}^{\mathcal{K}}$-quotient mappings are hereditarily $\mathcal{I}^{\mathcal{K}}$-quotient.*

Proof. Let $f\colon X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-quotient map and D is a subspace of Y. Consider $g = f|_{f^{-1}(D)}$ and the restriction map $g\colon f^{-1}(D) \to D$. Clearly, g is an $\mathcal{I}^{\mathcal{K}}$-continuous map. Consider an $\mathcal{I}$-nonthin sequence $(y_n)_{n\in L}$ in D, such that $(y_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to y in D. Since f is $\mathcal{I}^{\mathcal{K}}$-quotient map, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in T}$ with $x_{m_i} \in f^{-1}(y_{n_i}) \in f^{-1}(D)$, such that $(x_n)_{n\in T}$ $(\mathcal{I}|_T)^{\mathcal{K}}$-converges to $x \in f^{-1}(y) \in f^{-1}(D)$, where $T = \{m_1 < m_2 < \ldots\}$ and $\{n_1 < n_2 < \ldots\}$ are $\mathcal{I}$-nonthin subset of L. Therefore, g is an $\mathcal{I}^{\mathcal{K}}$-quotient map. $\square$

Example 3. Consider the space $X = [1, \omega_1]$ with the order topology, where ω_1 is the first uncountable ordinal and the space $Y = \{0, 1\}$ with topology $\{\varnothing, \{0\}, Y\}$. A function $f\colon X \to Y$ is defined by $f([1, \omega_1)) = \{0\}$ and $f(\omega_1) = 1$. Then f is a continuous quotient map. Again, no $\mathcal{I}$-nonthin sequence in $X\backslash\{\omega_1\}$ $\mathcal{I}^{\mathcal{K}}$-converges to ω_1. This implies that $[1,\omega_1)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. Therefore, the set $f^{-1}(\{0\}) = [1, \omega_1)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. But the set $\{0\}$ is not $\mathcal{I}^{\mathcal{K}}$-closed in Y. Hence, f is not an $\mathcal{I}^{\mathcal{K}}$-quotient map.

Example 4. Consider the space $X = [1, \omega_1]$ with the discrete topology and the space $Y = [1, \omega_1]$ with order topology. Let $f\colon X \to Y$ be the identity map. Then f is continuous but not a quotient map. Suppose $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in Y, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $x \in Y$. Then there exists $M \in \mathcal{F}(\mathcal{I}|_L)$, such that the sequence $(y_n)_{n\in L}$ given by $y_n = x_n$, $n \in M$, and $y_n = x$, if $n \in L\backslash M$ $\mathcal{K}$-converges to x. There exists an open set U_0 containing x, $y_n \notin U_0$ for each $y_n \neq x$, so

$\{n \in L\colon y_n \neq x\} = \{n \in L\colon y_n \notin U_0\} \in \mathcal{K}$. Therefore, $\{n \in L\colon y_n \notin \{x\}\} \in \mathcal{K}$ and thus $(x_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x in X. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map.

Theorem 10.

(a) *Suppose $f\colon X \to Y$ is an $\mathcal{I}^{\mathcal{K}}$-continuous quotient map and X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Then Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space and the map f is $\mathcal{I}^{\mathcal{K}}$-quotient.*

(b) *If $f\colon X \to Y$ is $\mathcal{I}^{\mathcal{K}}$-quotient and Y is $\mathcal{I}^{\mathcal{K}}$-sequential, then f is quotient.*

Proof. (a) Let G be an $\mathcal{I}^{\mathcal{K}}$-open set in Y. Suppose $(x_n)_{n\in L}$ is an $\mathcal{I}$-nonthin sequence in X, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x in $f^{-1}(G)$. Since f is $\mathcal{I}^{\mathcal{K}}$-continuous, $(f(x_n))_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to $f(x)$ in G. Again, since G is $\mathcal{I}^{\mathcal{K}}$-open, from Theorem 2 it follows that $|\{n \in L\colon f(x_n) \in G\}| = \omega$. Thus $|\{n \in L\colon x_n \in f^{-1}(G)\}| = \omega$. Therefore, $f^{-1}(G)$ is $\mathcal{I}^{\mathcal{K}}$-open in X. Now, let $H \subset Y$ and $f^{-1}(H)$ be $\mathcal{I}^{\mathcal{K}}$-open in X. As X is $\mathcal{I}^{\mathcal{K}}$-sequential, $f^{-1}(H)$ is open in X. Again, since f is a quotient map, H is open in Y. Therefore, H is $\mathcal{I}^{\mathcal{K}}$-open in Y. Hence, f is an $\mathcal{I}^{\mathcal{K}}$-quotient map.
(b) Suppose $U \subset Y$ and $f^{-1}(U)$ is open in X. Then $f^{-1}(U)$ is $\mathcal{I}^{\mathcal{K}}$-open in X. Since f is $\mathcal{I}^{\mathcal{K}}$-quotient, U is $\mathcal{I}^{\mathcal{K}}$-open in Y. Again, since Y is $\mathcal{I}^{\mathcal{K}}$-sequential, U is open in Y. Hence, f is a quotient map. □

Corollary 4. *Let X and Y be topological spaces. Suppose $g\colon X \to Y$ is a continuous function and X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Then g is quotient if and only if g is $\mathcal{I}^{\mathcal{K}}$-quotient and Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

4. $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Definition 8. *A topological space X is said to be $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn if for each $A \subset X$ and each $x \in cl(A)$, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in A $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point x.*

Every Fréchet-Urysohn space[8] is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. The disjoint topological sum of any family of $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn spaces is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Consider a nonempty subspace G of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space X and $x \in cl_G(D)$, where $D \subset G$. Then $cl_G(D) = G \cap cl_X(D)$ and, so, $x \in cl_X(D)$. Since X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in D $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point x. Therefore, subspace of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Theorem 11. *Every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Proof. Suppose U is an $\mathcal{I}^{\mathcal{K}}$-open subset of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space X. Let $l \in \overline{(X \backslash U)}$. Then there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $X\backslash U$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to l. Since $X\backslash U$ is $\mathcal{I}^{\mathcal{K}}$-closed, $l \in X\backslash U$. Therefore, $X\backslash U$ is a closed set. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. □

Corollary 5. *Every Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-sequential.*

Example 5 is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, which is sequential but not Fréchet-Urysohn.

Example 5. Consider the space $X = \{0\} \cup \bigcup\limits_{i=1}^{\infty} X_i$, $X_i = \{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{i^2}, \frac{1}{i} + \frac{1}{i^2+1}, \frac{1}{i} + \frac{1}{i^2+2}, \dots\}$. Then $X_i \cap X_k = \varnothing$, for $i \neq k$. A topology τ on X consists of each $\{\frac{1}{i} + \frac{1}{j}\}$ and for an element x of the form $\frac{1}{i}$, sets are given by $\{\frac{1}{i}\} \cup \{\frac{1}{i} + \frac{1}{k}, \frac{1}{i} + \frac{1}{k+1}, \dots\}$, for $k = i^2, i^2+1, \dots$ and sets containing 0 are obtained from X by removing a finite number of X_i's and a finite number of points in all of the remaining X_i's that have the form $\frac{1}{i} + \frac{1}{j}$ ([5], Example 1.6.19). Consider the ideals $\mathcal{I} = \mathcal{K} = \{A\colon A \cap \Delta_i$ are finite for all but finitely many i $\}$, where $\mathbb{N} = \bigcup\limits_{i=1}^{\infty} \Delta_i$ and each Δ_i is infinite and $\Delta_i \cap \Delta_j = \varnothing$, $i \neq j$.

Let $A \subset X$ and $a \in \bar{A}$. If $a = \frac{1}{i} + \frac{1}{j}$, then $a \in A$. If $a = \frac{1}{i}$, then there exists an infinite subset Y_i of X_i, such that $Y_i \subset A$. Consider a sequence (x_n) in A, where $x_n = \frac{1}{i} + \frac{1}{i^2+k_n}$, (k_n) is an increasing sequence of natural numbers. Therefore, (x_n) $\mathcal{I}^{\mathcal{K}}$-converges to a. If $a = 0$, then there exists an increasing sequence $C = (c_n)$ of natural numbers, such that $\bigcup\limits_{i\in C} Y_i \subset A$ and each Y_i is an infinite subset of X_i. For each $i \in C$, consider a sequence (x_j) in A, defined by $x_j = \frac{1}{i} + \frac{1}{i^2+l_{i,j}}$, $j \in \Delta_i$, and $(l_{i,j})$ is an increasing sequence of natural numbers. Then (x_j) $\mathcal{I}^{\mathcal{K}}$-converges to 0. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Moreover, from Example 1.6.19 in [5], X is sequential but not Fréchet-Urysohn.

Example 6. Let $S = (a_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements. Consider the space $X = S \cup \{\alpha\}$, $\alpha \notin S$. A topology τ on X consists of each $\{a_n\}$ and sets U containing α of the form $U = \{\alpha\} \cup \{a_n \colon n \in L\}$, where $\mathbb{N}\backslash L \in \mathcal{K}$.

Let $A \subset X$ and $a \in \bar{A}$. If $a \in S$, then $a \in A$. Then, taking the constant sequence (a), space becomes $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. If $a = \alpha$ and $a \notin A$, then A is a subset of S. Assume that A is a $\mathcal{K}$-thin subsequence of S. Then

$X \backslash A = U$ is an open neighborhood of $\alpha = a$. But $a \in \bar{A}$ and $A \cap U = \varnothing$, which leads to a contradiction. Therefore, A is a $\mathcal{K}$-nonthin subsequence of S. So, A $\mathcal{K}$-converges to a and, then, A $\mathcal{I}^{\mathcal{K}}$-converges to a. Therefore, X is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn. Again, by Theorem 11, X is $\mathcal{I}^{\mathcal{K}}$-sequential. It is obvious that $\alpha \in \bar{S}$. Let $(a_n)_{n \in L}$ be a subsequence of S. Consider a $\mathcal{K}$-thin subsequence $(a_n)_{n \in L_1}$ of $(a_n)_{n \in L}$. Let $U = X \backslash \{a_n : n \in L_1\}$. Then U is an open neighborhood of α. Therefore, $(a_n)_{n \in L}$ does not converge to α. So, no subsequence of S converges to α. Hence, X is not a Fréchet-Urysohn space.

Nowhere tall ideal plays an important role in the following theorem. An ideal $\mathcal{I}$ on a non-empty set X is nowhere tall if for any set $A \notin \mathcal{I}$, there exists $B \subset A$, such that $\mathcal{I}|_B$ is the collection of all finite subsets of B ([7], Definition 2.25).

Theorem 12. *$\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is Fréchet-Urysohn provided $\mathcal{K} \subset \mathcal{I}$ and $\mathcal{K}$ is a nowhere tall ideal on $\mathbb{N}$.*

Proof. Let X be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, $A \subset X$ and $a \in \bar{A}$. There exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n \in L}$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to a. Then there is a set $M \in \mathcal{F}(\mathcal{I}|_L)$, such that the sequence $(y_n)_{n \in L}$ given by $y_n = x_n$, $n \in M$, and $y_n = a$, $n \in L \backslash M$ $\mathcal{K}|_L$-converges to a. Since $\mathcal{K} \subset \mathcal{I}$ and $M \notin \mathcal{I}$, $M \notin \mathcal{K}$. As $\mathcal{K}$ is a nowhere tall ideal, there exists a subset M_1 of M, such that $\mathcal{I}|_{M_1}$ is the collection of all finite subsets of M_1. Therefore, the sequence $(x_n)_{n \in M_1}$ converges to a and, so, X is a Fréchet-Urysohn space. □

Example 7. For each $i \in \mathbb{N}$, consider a sequence of distinct elements $S_i = \{x_{i,j} : j \in \mathbb{N}\}$. Let $S = \{a_i : i \in \mathbb{N}\}$ be a sequence of distinct elements. Consider the space $X = \cup\{S_i : i \in \mathbb{N}\} \cup S \cup \{\alpha\}$, $\alpha \notin \cup\{S_i : i \in \mathbb{N}\} \cup S$. A topology τ on X consists of each $\{x_{i,j}\}$ and sets containing a_i of the form $\{a_i\} \cup \{x_{i,j} : j \in T\}$, $\mathbb{N} \backslash T \in \mathcal{K}$ for each $i \in \mathbb{N}$, and sets containing α of the form $\{\alpha\} \cup \{a_i : i \in L\} \cup \{\{x_{i,j} : j \in T\} : i \in L\}$ for each $i \in \mathbb{N}$, where $\mathbb{N} \backslash L \in \mathcal{K}$ and $\mathbb{N} \backslash T \in \mathcal{K}$.

Consider an $\mathcal{I}^{\mathcal{K}}$-closed subset Y of X. Let $p \in \bar{Y}$. If $p \in \bigcup_{i=1}^{\infty} S_i$, $\{p\}$ is an open set. As $p \in \bar{Y}$, $p \in Y$. If $p = \alpha$, consider the subsequence $Y \cap S$ of S. Since $\alpha \in \bar{Y}$, $Y \cap S$ is a $\mathcal{K}$-nonthin subsequence and, so, $Y \cap S$ $\mathcal{I}^{\mathcal{K}}$-converges to α. Therefore, $p \in \overline{Y}^{\mathcal{I}^{\mathcal{K}}} = Y$. If $p \in S$, there exists $i_0 \in \mathbb{N}$, such that $a_{i_0} = p$. Consider the subsequence $Y \cap S_{i_0}$ of S_{i_0}. Since $p \in \bar{Y}$, $Y \cap S_{i_0}$ is a $\mathcal{K}$-nonthin subsequence of S_{i_0}. Therefore $Y \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges

to a_{i_0}. So, $p = a_{i_0} \in \overline{Y}^{\mathcal{I}^{\mathcal{K}}} = Y$. Hence, Y is a closed subset of X. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-sequential space.
It is obvious that X is Hausdorff and $\alpha \in \overline{X\backslash(S \cup \{\alpha\})}$. Let $E = (y_n)_{n\in L}$ be a sequence in $X\backslash(S \cup \{\alpha\})$, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converge to α. If for each $i \in \mathbb{N}$, $E_i = E \cap S_i$ is a $\mathcal{K}$-thin sequence of S_i, then take $U = \{\alpha\} \cup S \cup \{S_i\backslash E_i\colon i \in \mathbb{N}\}$. Then U is an open set containing α and $U \cap E = \varnothing$, which leads to a contradiction. Therefore, there exists $i_0 \in \mathbb{N}$, such that $E \cap S_{i_0}$ is a $\mathcal{K}$-nonthin subsequence of S_{i_0}. So $E \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges to $a_{i_0} \neq \alpha$. Again, by assumption $E \cap S_{i_0}$ $\mathcal{I}^{\mathcal{K}}$-converges to α, which leads to a contradiction as X is Hausdorff. Therefore, no sequence in $X\backslash(S \cup \{\alpha\})$ $\mathcal{I}^{\mathcal{K}}$-converges to α. Hence, X is not an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Theorem 13. *A topological space X is hereditarily $\mathcal{I}^{\mathcal{K}}$-sequential if and only if the space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn.*

Proof. Suppose $G \subset X$ and $x \in \bar{G}$. Without loss of generality, let $x \notin G$. Then G is not a closed set in X. Let $Y = G \cup \{x\}$. Therefore, G is not closed in Y. As Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space, G is not an $\mathcal{I}^{\mathcal{K}}$-closed set in Y. There exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in G, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to x. Hence, X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Conversely let X be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Again, subspace of an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn and every $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space is $\mathcal{I}^{\mathcal{K}}$-sequential. Therefore, the space X is hereditarily $\mathcal{I}^{\mathcal{K}}$-sequential. $\square$

A mapping $f\colon X \to Y$ is said to be pseudo-open if for each $p \in Y$ and each neighbourhood O of $f^{-1}(p)$ in X, $p \in int(f(O))$ [5].

Theorem 14. *Let X, Y be topological spaces and let Y be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Then each $\mathcal{I}^{\mathcal{K}}$-covering mapping f from X onto Y is pseudo-open.*

Proof. Suppose f is not a pseudo-open map. Then there exists a point $z \in Y$ and an open subset O of X, such that $f^{-1}(z) \subset O$ and z is not an interior point of $f(O)$. So, $z \in \overline{Y\backslash f(O)}$. As Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $(z_n)_{n\in L}$ in $Y\backslash f(O)$, such that $(z_n)_{n\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to z. Also, since f is an $\mathcal{I}^{\mathcal{K}}$-covering mapping, there exists an $\mathcal{I}$-nonthin sequence $(\alpha_i)_{i\in L}$ with $\alpha_i \in f^{-1}(z_i)$, for all $i \in L$ and $\alpha \in f^{-1}(z)$, such that $(\alpha_i)_{i\in L}$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converges to α. Therefore, $\alpha \in O$ and $\{i \in L\colon \alpha_i \notin O\} \in \mathcal{K}|_L$. Then there exists $t \in L$, such that $\alpha_t \in O$ and, so, $z_t \in f(O)$, which leads to a contradiction. Hence, f is a pseudo-open map. $\square$

Theorem 15. *Suppose $f\colon X \to Y$ is a quotient map, where X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Then Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space if and only if f is pseudo open.*

Proof. Let $G \subset Y$ and $p \in \bar{G}$. If possible, let $f^{-1}(p) \cap \overline{f^{-1}(G)} = \varnothing$. Then $f^{-1}(p) \subset X \backslash \overline{f^{-1}(G)} = O$ (say). As f is pseudo-open, then $p \in int f(O)$. Again, $int f(O) \subset int f(X \backslash f^{-1}(G)) = int(Y \backslash G) = Y \backslash \bar{G}$. Thus $p \in Y \backslash \bar{G}$, which leads to a contradiction. Therefore, there exists a point $q \in f^{-1}(p) \cap \overline{f^{-1}(G)}$. Since X is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, there exists an $\mathcal{I}$-nonthin sequence $(x_n)_{n\in L}$ in $f^{-1}(G)$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to the point q. Therefore, there exists an $\mathcal{I}$-nonthin sequence $(f(x_n))_{n\in L}$ in G, which $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to $f(q) = p$. Hence, Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space.

Conversely let Y be an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space. Suppose $p \in Y$ and O is an open neighbourhood of $f^{-1}(p)$. Let us assume that $p \notin int f(O)$. Then $p \in \overline{Y \backslash f(O)}$. Since Y is an $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn space, there exists an $\mathcal{I}$-nonthin sequence $S = (y_n)_{n\in L}$ in $Y \backslash f(O)$ $(\mathcal{I}|_L)^{\mathcal{K}}$-converging to p. Again, since f is a quotient map, $\overline{f^{-1}(S)} \subset f^{-1}(\bar{S}) = f^{-1}(S) \cup f^{-1}(p)$. Since O is an open neighborhood of $f^{-1}(p)$ and $O \cap f^{-1}(S) = \varnothing$, $f^{-1}(p) \cap \overline{f^{-1}(S)} = \varnothing$ and so $f^{-1}(S)$ is closed. Therefore, $X \backslash f^{-1}(S) = f^{-1}(Y \backslash S)$ is open. Since f is quotient, $Y \backslash S$ is open, which leads to a contradiction. Hence $p \in int f(O)$ and so f is pseudo open. □

The article is concluded with the diagram (Figure 1), which shows interrelations among Fréchet-Urysohn, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, sequential, and $\mathcal{I}^{\mathcal{K}}$-sequential spaces.

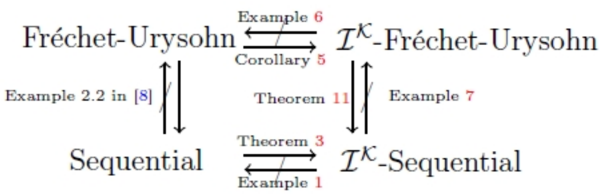

Figure 1: Relation among Fréchet-Urysohn, $\mathcal{I}^{\mathcal{K}}$-Fréchet-Urysohn, sequential, and $\mathcal{I}^{\mathcal{K}}$-sequential spaces

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