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WEAVING G-FUSION FRAMES IN HILBERT SPACES

Abstract. Fusion frames, one of the important frame extensions, introduced by Casazza and Kutyniok, Bemros et al. are able to transform the except of weaving from Gabor frames to frame theory; thus weaving frames. In this paper, we generalize the weaving fusion frames and study them from the viewpoint of the finite and infinite dimensional Hilbert spaces. Also, we introduce weakly g-fusion woven frame and show that they are equivalent with the weaving g-fusion frames.

Key words: *Hilbert spaces, g-fusion frame, woven frames, gf-Riesz sequence*

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1. Introduction and preliminaries. During the last 20 years, the theory of frames has been growing rapidly, especially for the reason that several new applications have been developed. For example, besides traditional applications, such as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission, and to design high-rate constellations with full diversity in multiple-antenna code design (e.g. [7], [11], [15], [13]). Woven frames have been introduced in [1] and they have potential applications in wireless sensor networks. Afterwards, this topic was presented in other frames like g-frames, fusion frames, etc [17], [22], [23]. Recently, authors were able to generalize fusion frames; this new object was called *g-fusion frames* in Hilbert spaces ([20], [18]). In this note, we aim at studying woven version for these frames.

Throughout this paper, H and K are separable Hilbert spaces and $\mathcal{B}(H, K)$ is the collection of all the bounded linear operators of H into K. If $H = K$, then we use the notation $\mathcal{B}(H)$ instead of $\mathcal{B}(H, H)$. Also, π_V

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is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. Finally, take $[m] := \{1, 2, \dots, m\}$ for each $m > 1$.

In this section, we review some basic concepts and some results.

Lemma 1. *[12] Let $V \subseteq H$ be a closed subspace, and U be a linear bounded operator on H. Then*

$$\pi_V U^* = \pi_V U^* \pi_{\overline{UV}}.$$

If U is a unitary operator (U is bijective and $U^* = U^{-1}$), then $\pi_{\overline{UV}} U = U \pi_V$.

Lemma 2. *[Open Mapping Theorem] [16] A bounded linear operator U from a Banach space X onto a Banach space Y is an open mapping. Hence, if U is bijective, U^{-1} is continuous and thus bounded.*

Definition 1. *[woven frame] [1] A family of frames $\{f_{ij}\}_{i\in[m],\, j\in\mathbb{J}}$ for H is said to be woven if there are universal constants A and B, such that for each partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $\{f_{ij}\}_{i\in[m],\, j\in\sigma_i}$ is a frame for H with lower and upper frame bounds A and B, respectively. Each family $\{f_{ij}\}_{i\in[m],\, j\in\sigma_i}$ is called a weaving.*

We call A and B the lower and upper frame bounds, respectively. The optimal upper (lower) frame bound is the infimum (supremum) over all upper (lower) frame bounds. When $A = B = 1$, then $\{f_{ij}\}_{i\in[m],\, j\in\mathbb{J}}$ is called a Parseval woven frame. It is easy to see that every weaving has an universal upper frame bound by the sum of the upper frame bounds of $\{f_{ij}\}_{i\in[m],\, j\in\mathbb{J}}$. So, we only need to check that there is an universal lower frame bound for all weavings. But we notice that this universal upper bound cannot be the smallest upper weaving. Indeed, if B_i are optimal upper bounds of $\{f_{ij}\}_{i\in[m],\, j\in\mathbb{J}}$, then $\sum_{i\in[m]} B_i$ may not be optimal. Also, if $\{f_{ij}\}_{i\in[m],\, j\in\mathbb{J}}$'s are Parseval frames, but the constructing woven by them may not be Parseval. For this, let $\varepsilon > 0$ and $\delta = (1+\varepsilon^2)^{-\frac{1}{2}}$. Assume that $\{e_1, e_2\}$ is the standard orthonormal basis for $\mathbb{R}^2$ and, also,

$$\{f_j\}_{j=1}^4 := \{\delta e_1, \delta\varepsilon e_1, \delta e_2, \delta\varepsilon e_2\}, \qquad \{g_j\}_{j=1}^4 := \{\delta\varepsilon e_1, \delta e_1, \delta\varepsilon e_2, \delta e_2\}.$$

Then $\{f_j\}_{j=1}^4$ and $\{g_j\}_{j=1}^4$ are Parseval frames and, also, they are woven frames with the universal upper frame bound 2 (for more details, we refer to [1]).

We define the space $\mathscr{H}_2 := (\sum_{j\in\mathbb{J}} \oplus H_j)_{\ell_2}$ by

$$\mathscr{H}_2 = \Big\{\{f_j\}_{j\in\mathbb{J}} \colon f_j \in H_j,\ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\Big\}. \tag{1}$$

With the inner product defined by

$$\langle \{f_j\}, \{g_j\}\rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j\rangle,$$

it is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations.

Definition 2. *[g-fusion frame] [20] Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a collection of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i.e., $v_j > 0$, $\Lambda_j \in \mathcal{B}(H, H_j)$ for each $j \in \mathbb{J}$. We say $\Lambda := (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is a g-fusion frame for H if there exist $0 < A \leqslant B < \infty$, such that for each $f \in H$*

$$A\|f\|^2 \leqslant \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leqslant B\|f\|^2. \tag{2}$$

When the right-hand side of (2) holds, Λ is called to be a g-fusion Bessel sequence for H with the bound B. We say Λ is a Parseval g-fusion frame whenever $A = B = 1$. The synthesis and the analysis operators for a g-fusion frame are defined as follows (for more details, refer to [20]):

$$\begin{cases} T_\Lambda \colon \mathscr{H}_2 \longrightarrow H, \\ T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j, \end{cases} \quad \text{and} \quad \begin{cases} T_\Lambda^* \colon H \longrightarrow \mathscr{H}_2, \\ T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}}. \end{cases}$$

The g-fusion frame operator is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f,$$

and

$$\langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2, \tag{3}$$

for all $f \in H$. Therefore,

$$A Id_H \leqslant S_\Lambda \leqslant B Id_H. \tag{4}$$

Hence, S_Λ is a bounded, positive, and invertible operator (with the adjoint inverse). A g-fusion frame $\widetilde{\Lambda} := (S_\Lambda^{-1} W_j, \Lambda_j \pi_{W_j} S_\Lambda^{-1}, v_j)_{j\in\mathbb{J}}$ with the g-fusion frame operator $S_{\widetilde{\Lambda}} = T_{\widetilde{\Lambda}} T^*_{\widetilde{\Lambda}}$ is called the *(canonical) dual g-fusion frame* of Λ. Now, we can obtain

$$f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \widetilde{\Lambda_j} \pi_{\widetilde{W_j}} f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{\widetilde{W_j}} \widetilde{\Lambda_j}^* \Lambda_j \pi_{W_j} f, \tag{5}$$

where $\widetilde{W_j} := S_\Lambda^{-1} W_j$, $\widetilde{\Lambda_j} := \Lambda_j \pi_{W_j} S_\Lambda^{-1}$.

Definition 3. *[18] $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is called a gf-Riesz basis for H if*

1) *Λ is gf-complete, i.e., $\overline{\text{span}}\{\pi_{W_j}\Lambda_j^* H_j\} = H$,*
2) *There exist $0 < A \leqslant B < \infty$, such that for each finite subset $\mathbb{I} \subseteq \mathbb{J}$, $g_j \in H_j$ and $j \in \mathbb{I}$,*

$$A \sum_{j\in\mathbb{I}} \|g_j\|^2 \leqslant \Big\| \sum_{j\in\mathbb{I}} v_j \pi_{W_j} \Lambda_j^* g_j \Big\|^2 \leqslant B \sum_{j\in\mathbb{I}} \|g_j\|^2. \tag{6}$$

When only item 2) is valid, we say that Λ is a gf-Riesz sequence. It is easy to check that every subfamily of a gf-Riesz sequence is a gf-Riesz sequence.

Lemma 3. *[18] Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for H. Then the following are equivalent:*

(I) *Λ is a gf-Riesz basis for H.*

(II) *For any finite subset $\mathbb{I} \subset \mathbb{J}$ if $\sum_{j\in\mathbb{I}} v_j \pi_{W_j} \Lambda_j^* g_j = 0$ for some $\{g_j\}_{j\in\mathbb{I}} \in \mathscr{H}_2$, then $g_j = 0$ for all $j \in \mathbb{J}$.*

If a g-fusion frame is not a gf-Riesz basis, it is said to be *gf-overcomplete*. Now by Lemma 3, we can set a remark:

Remark 1. *if $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ is gf-overcomplete, then there exist a finite subset $\mathbb{I} \subset \mathbb{J}$ and $\{g_j\}_{j\in\mathbb{I}} \in \mathscr{H}_2 \backslash \{0\}$ for which*

$$\sum_{j\in\mathbb{I}} v_j \pi_{W_j} \Lambda_j^* g_j = 0.$$

2. Weaving g-fusion frames. Throughout this paper, $\{W_{ij}\}_{j\in\mathbb{J},\, i\in[m]}$ is a collection of closed subspaces of H, $\{v_{ij}\}_{j\in\mathbb{J},\, i\in[m]}$ is a family of weights, and $\{\Lambda_{ij}\}_{j\in\mathbb{J},\, i\in[m]} \subseteq \mathcal{B}(H, H_{ij})$.

Definition 4. *A family of g-fusion frames $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ for H is said to be g-fusion woven frame if there exist universal positive constants A and B, such that for each partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i,\, i\in[m]}$ is a g-fusion frame for H with bounds A and B.*

Now we show that every g-fusion woven frame has a universal upper frame bound and (see Corollary 6) we will show that it may not be optimal.

Proposition 1. *Let $\{(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}\}$ be a g-fusion Bessel sequence for H with the bound B_i for each $i \in [m]$. Then, for any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i, i\in[m]}$ is a g-fusion Bessel sequence with the Bessel bound $\sum_{i\in[m]} B_i$.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of $\mathbb{J}$. For each $f \in H$, we have

$$\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \leqslant \Big(\sum_{i\in[m]} B_i\Big)\|f\|^2.$$

□

In the following results, we construct a g-fusion woven frame by using a bounded linear operator.

Theorem 1. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a g-fusion woven frame for H with the universal bounds A and B and $U \in \mathcal{B}(H, K)$. Then $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is a g-fusion woven frame for K if and only if there exists a number $\delta > 0$, such that for every $f \in K$:*

$$\|U^* f\| \geqslant \delta\|f\|.$$

Proof. Let $f \in K$ and $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a g-fusion woven frame for K with the lower bound C and $U \in \mathcal{B}(H, K)$. So, by Lemma 1, we get

$$C\|f\|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*\pi_{\overline{UW_{ij}}}f\|^2 = \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^* f\|^2.$$

On the other hand, we have

$$\sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^* f\|^2 \leqslant B\|U^* f\|^2.$$

Thus, $\|U^* f\| \geqslant \sqrt{\frac{C}{B}}\|f\|$. For the opposite implication, we can write for all $f \in K$:

$$A\delta^2\|f\|^2 \leqslant A\|U^* f\|^2 \leqslant \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^* f\|^2$$

$$= \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}U^*\pi_{\overline{UW_{ij}}}f\|^2 \leqslant B\|U\|^2\|f\|^2.$$

So, $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is a g-fusion woven frame for K with frame bounds $A\delta^2$ and $B\|U\|^2$. □

Corollary 1. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ be a g-fusion woven frame for H with common frame bounds A and B, and assume that $U\in\mathcal{B}(H)$ has closed range. Then $(\overline{UW_{ij}}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is also g-fusion woven frame for $\mathcal{R}(U)$ with frame bounds $A\|U^\dagger\|^{-2}$ and $B\|U\|^2$.*

Corollary 2. *In Corollary 1, if $U\in\mathcal{B}(H)$ is an invertible operator, then, by Lemmas 2, we have: UW_{ij} is closed for any $j\in\mathbb{J}$ and therefore $(UW_{ij}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is a g-fusion woven frame for H with frame bounds $A\|U^{-1}\|^{-2}$ and $B\|U\|^2$. In particular, if U is also unitary, then the bounds of $(UW_{ij}, \Lambda_{ij}\pi_{W_{ij}}U^*, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ are the same with bounds of $(W_{ij},\Lambda_{ij}, v_{ij})$.*

Corollary 3. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ be a g-fusion woven frame for H with frame operator S^i_Λ for $i\in[m]$. Then the canonical dual $\big((S^i_\Lambda)^{-1}W_{ij}, \Lambda_{ij}\pi_{W_{ij}}(S^i_\Lambda)^{-1}, v_{ij}\big)_{j\in\mathbb{J},\, i\in[m]}$ is also a g-fusion woven frame for H.*

Theorem 2. *Let $(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for H with bounds A and B and $U\in\mathcal{B}(H)$ be a unitary operator. If $\|Id_H - U\|^2 \leqslant \dfrac{A}{B}$, then $\{(W_j, \Lambda_j, v_j)_{j\in\mathbb{J}}, (U^{-1}W_j, \Lambda_jU, v_j)_{j\in\mathbb{J}}\}$ is a g-fusion woven frame for H.*

Proof. By Proposition 1 and corollary 2, the upper bound is clear. Let $\sigma\subset\mathbb{J}$ be a partition and $f\in H$. So, by Lemma 1, we can write

$$\sum_{j\in\sigma} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \sum_{j\in\sigma^c} v_j^2\|\Lambda_jU\pi_{U^{-1}W_j}f\|^2$$
$$= \sum_{j\in\sigma} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \sum_{j\in\sigma^c} v_j^2\|\Lambda_j\pi_{W_j}f - (\Lambda_j\pi_{W_j}f - \Lambda_j\pi_{W_j}Uf)\|^2$$
$$\geqslant \sum_{j\in\mathbb{J}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \sum_{j\in\sigma^c} v_j^2\|\Lambda_j\pi_{W_j}(Id_H - U)f\|^2 \geqslant \big(A - B\|Id_H - U\|\big)\|f\|^2.$$

Thus, $(W_j, \Lambda_j, v_j)_{j\in\sigma}\bigcup(U^{-1}W_j, \Lambda_jU, v_j)_{j\in\sigma^c}$ is a g-fusion frame. □

Proposition 2. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ be a g-fusion woven frame for H with common frame bounds A and B. Let $0 < C \leqslant |\omega_j^{(i)}|^2 \leqslant D < \infty$ for any $i\in[m]$ and $j\in\mathbb{J}$; then $(W_{ij}, \omega_j^{(i)}\Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is a g-fusion woven frame for H with frame bounds AC and BD.*

Proof. For any partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$ and $f \in H$, we get

$$AC\|f\|^2 = \min_{i\in[m]} |\omega_j^{(i)}|^2 A\|f\|^2 \leqslant \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \|\omega_j^{(i)} \Lambda_{ij} \pi_{W_{ij}} f\|^2$$
$$\leqslant \max_{i\in[m]} |\omega_j^{(i)}|^2 B\|f\|^2 = BD\|f\|^2.$$

This implies the statement. □

Theorem 3. *Let $\mathbb{I} \subset \mathbb{J}$ be non-empty and $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{I},\, i\in[m]}$ be a g-fusion woven frame for H. Then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is a g-fusion woven frame.*

Proof. Assume that $\sigma_i \subset \mathbb{J}$. By Proposition 1, we know that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}$ is a g-fusion Bessel sequence for H for any $i \in [m]$. If A is the lower bound of $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i\cap\mathbb{I},\, i\in[m]}$, then for every $f \in H$ we have

$$A\|f\|^2 \leqslant \sum_{i\in[m]} \sum_{j\in\sigma_i\cap\mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 \leqslant \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2.$$

This implies the statement. □

The next theorem shows that even if one subspace is deleted, it still remains a g-fusion woven frame.

Theorem 4. *Let $\mathbb{I}$ be an infinite subset of $\mathbb{I} \subset \mathbb{J}$ and $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a g-fusion woven frame for H with the bounds A and B. If*

$$C := \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\|^2 < \infty,$$

then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}\setminus\mathbb{I},\, i\in[m]}$ is also a g-fusion woven frame for H with frame bounds $A - C$ and B.

Proof. The upper bound is obvious. Suppose that $\{\sigma_i\}_{i\in[m]} \subset \mathbb{J}\setminus\mathbb{I}$ and $f \in H$, so we can write

$$\sum_{i\in[m]} \sum_{j\in\sigma_i} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 = \sum_{i\in[m]} \sum_{j\in\sigma_i \bigcup \mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 - \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2$$
$$\geqslant A\|f\|^2 - \sum_{i\in[m]} \sum_{j\in\mathbb{I}} v_{ij}^2 \|\Lambda_{ij}\|^2 \|f\|^2 = (A - C)\|f\|^2.$$

□

In the following example, we show that the infinite condition for the set $\mathbb{J}$ is necessary in Theorem 4.

Example 1. We attend to the Hilbert space $H := \mathbb{R}^3$ with standard base $\{e_1, e_2, e_3\}$. Let $\mathbb{J} = \{1, 2, 3\}$ and

$$\begin{aligned} W_1 = \operatorname{span}\{e_1, e_2\}, \quad W_2 = \operatorname{span}\{e_2, e_3\}, \quad W_3 = \operatorname{span}\{e_3, e_1\}, \\ X_1 = \operatorname{span}\{e_1\}, \quad X_2 = \operatorname{span}\{e_2\}, \quad X_3 = \operatorname{span}\{e_3\}, \\ v_1 = v_2 = v_3 = 1. \end{aligned}$$

Assume that $\Lambda_j, \Theta_j \colon H \to \mathbb{C}$ for any $j \in \mathbb{J}$ such that for each $f \in H$,

$$\begin{aligned} \Lambda_1 f &= \langle f, e_1\rangle, \quad \Lambda_2 f = \langle f, e_2\rangle, \quad \Lambda_3 f = \langle f, e_3\rangle, \\ \Theta_1 f &= \langle f, e_1 + e_2\rangle, \quad \Theta_2 f = \langle f, e_2 + e_3\rangle, \quad \Theta_3 f = \langle f, e_3 + e_1\rangle. \end{aligned}$$

It is easy to compute that $(W_j, \Lambda_j, 1)_{j\in\mathbb{J}}$ and $(X_j, \Theta_j, 1)_{j\in\mathbb{J}}$ are Parseval g-fusion woven frame for H. But if we choose $\mathbb{I} = \{1\}$, then the deletion of this index of subspaces destroys the g-fusion woven frame property.

Corollary 4. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ be a tight g-fusion woven frame for H with the bound A. Assume that $j_0 \in \mathbb{J}$. Then the following stalemates are equivalent:*
(I) $\sum_{i\in[m]} v_{ij_0}^2 \|\Lambda_{ij_0}\pi_{W_{ij_0}}\|^2 < A$;
(II) $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}\setminus\{j_0\}, i\in[m]}$ *is a g-fusion woven frame for H.*

Proof. $(I) \Rightarrow (II)$ is clear by Theorem 4. For the opposite implication, suppose that C and D are the frame bounds of $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}\setminus\{j_0\}, i\in[m]}$. For any $0 \neq f \in H$, we have

$$\begin{aligned} C\|f\|^2 &\leqslant \sum_{i\in[m]} \sum_{j\in\mathbb{J}\setminus\{j_0\}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 \\ &= \sum_{i\in[m]} \sum_{j\in\mathbb{J}} v_{ij}^2 \|\Lambda_{ij}\pi_{W_{ij}} f\|^2 - \sum_{i\in[m]} v_{ij_0}^2 \|\Lambda_{ij_0}\pi_{W_{ij_0}} f\|^2 \\ &= A\|f\|^2 - \sum_{i\in[m]} v_{ij_0}^2 \|\Lambda_{ij_0}\pi_{W_{ij_0}} f\|^2 \leqslant D\|f\|^2. \end{aligned}$$

Hence,

$$0 < C \leqslant A - \sum_{i\in[m]} v_{ij_0}^2 \frac{\|\Lambda_{ij_0}\pi_{W_{ij_0}} f\|^2}{\|f\|^2} \leqslant D.$$

So, we conclude that $A-\sum_{i\in[m]} v_{ij_0}^2\|\Lambda_{ij_0}\pi_{W_{ij_0}}\|^2>0$. □

Theorem 5. *Let $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ be a g-fusion woven frame for H with the bounds A and B. For each $i\in[m]$, $j\in\mathbb{J}$ and an index set $\mathbb{I}_{ij}$, suppose that $\{f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}}\in\Lambda_{ij}(W_{ij})$ is a Parseval frame for H_{ij}, such that for every finite subset $\mathbb{K}_{ij}$ of $\mathbb{I}_{ij}$, the set $\{f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}$ is a frame with the lower bound C_{ij}. Then assume that $C:=\inf_{j\in\mathbb{J},i\in[m]}C_{ij}>0$. If $\widetilde{W}_{ij}:=\overline{span}\{\Lambda_{ij}^*f_{ij}^{(k)}\}_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}$ for any $i\in[m]$ and $j\in\mathbb{J}$, then $(\widetilde{W}_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is a g-fusion woven frame for H with the bounds CA and B.*

Proof. Obviously, B is an upper bound of $(\widetilde{W}_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},\, i\in[m]}$. Now, for considering the lower bound condition, we can write for each $f\in H$ and $\{\sigma_i\}_{i\in[m]}\in\mathbb{J}$ the following relations:

$$\begin{aligned}
&\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{\widetilde{W}_{ij}}f\|^2=\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\sum_{k\in\mathbb{I}_{ij}}|\langle\Lambda_{ij}\pi_{\widetilde{W}_{ij}}f,f_{ij}^{(k)}\rangle|^2\\
&\geqslant\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\sum_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}|\langle\pi_{\widetilde{W}_{ij}}f,\Lambda_{ij}^*f_{ij}^{(k)}\rangle|^2=\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\sum_{k\in\mathbb{I}_{ij}\setminus\mathbb{K}_{ij}}|\langle\Lambda_{ij}\pi_{W_{ij}}f,f_{ij}^{(k)}\rangle|^2\\
&\geqslant\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2C_{ij}\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\geqslant C\sum_{i\in[m]}\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\geqslant CA\|f\|^2.
\end{aligned}$$

□

3. Weakly g-fusion woven frames. In [1], Bemrose et al. the authors showed that if each weaving is a frame, then there exists a universal lower frame bound for all the weaving. For this, they introduced a special kind of weaving named *weakly woven*. After that, this topic has been studied for g-frames in [17]. Here we consider these results for g-fusion frames.

Definition 5. *A family of g-fusion frames $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{N},\, i\in[m]}$ in H is called weakly woven if $\forall\ \{\sigma_i\}_{i\in[m]}$ of $\mathbb{N}$, the family $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\sigma_i,\, i\in[m]}$ is a g-fusion frame.*

Similarly to Theorem 4.1 and Theorem in [1], [17], we can show that weakly woven is equivalent to woven in the finite case:

Theorem 6. *A finite family of g-fusion frames $(W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J},i\in[m]}$ for a finite-dimensional Hilbert space H^n is woven if and only if for each partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}$, $\{\pi_{ij}\Lambda_{ij}^*H_{ij}\}_{j\in\sigma_i,i\in[m]}$ spans the space.*

But for a infinite-dimensional case, their equivalence is more difficult to establish, similarly to frames and g-frames. The following theorem will be used in this case, which is a general case of Lemma 4.3 in [1]:

Theorem 7. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}$ be a g-fusion frame for H for each $i \in [m]$. Suppose that a collection of disjoint finite sets $\{\tau_i\}_{i\in[m]}$ of $\mathbb{J}$ and for any $\varepsilon > 0$ there exists a partition $\{\sigma_i\}_{i\in[m]}$ of $\mathbb{J}\backslash\bigcup_{i\in[m]}\tau_i$, such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_i\bigcup\tau_i),\, i\in[m]}$ has a lower g-fusion frame bound less than ε. Then $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ is not a woven frame.*

Proof. We can write $\mathbb{J} = \bigcup_{j\in\mathbb{N}}\mathbb{J}_j$, where $\mathbb{J}_j$'s are disjoint index sets. Assume that $\tau_{1j} = \varnothing$ for all $j \in [m]$ and $\varepsilon = 1$. Then there exists a partition $\{\sigma_{i1}\}_{i\in[m]}$ of $\mathbb{J}$, such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_{i1}\bigcup\tau_{i1}),\, i\in[m]}$ has a lower bound (also, optimal lower bound) less than 1. Thus, there is a vector $f_1 \in H$ with $\|f_1\| = 1$, such that

$$\sum_{i\in[m]}\sum_{j\in(\sigma_{i1}\bigcup\tau_{i1})} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_1\|^2 < 1.$$

Since

$$\sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_1\|^2 < \infty,$$

there exists $k_1 \in \mathbb{N}$, such that

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_1} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_1\|^2 < 1,$$

where $\mathbb{K}_1 = \bigcup_{i\geqslant k_1+1}\mathbb{J}_j$. By induction, for $\varepsilon = \frac{1}{n}$ and a partition $\{\tau_{nj}\}_{i\in[m]}$ of $\mathbb{J}_1\bigcup\cdots\bigcup\mathbb{J}_{k_n-1}$, such that

$$\tau_{ni} = \tau_{(n-1)i}\bigcup\left(\sigma_{(n-1)i}\bigcap(\mathbb{J}_1\bigcup\cdots\bigcup\mathbb{J}_{k_n-1})\right)$$

for all $i \in [m]$, there exists a partition $\{\sigma_{ni}\}_{i\in[m]}$ of $\mathbb{J}\backslash(\mathbb{J}_1\bigcup\cdots\bigcup\mathbb{J}_{k_n-1})$, such that $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_{ni}\bigcup\tau_{ni}),i\in[m]}$ has a lower bound less than $\frac{1}{n}$. Therefore, there is an element f_n of H and $k_n \in \mathbb{N}$, such that $\|f_n\| = 1$, $k_n > k_{n-1}$ and

$$\sum_{i\in[m]}\sum_{j\in\mathbb{K}_n} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f_n\|^2 < \frac{1}{n},$$

where $\mathbb{K}_n = \bigcup_{i\geqslant k_n+1}\mathbb{J}_j$. Choose a partition $\{\varsigma_i\}_{i\in[m]}$ of $\mathbb{J}$, where $\varsigma_i := \bigcup_{j\in\mathbb{N}}\{\tau_{ji}\} = \tau_{(n+1)i}\bigcup(\varsigma_i \cap \mathbb{J}\backslash(\mathbb{J}_1\bigcup\cdots\bigcup\mathbb{J}_n))$. Assume that $(W_{ij}, \Lambda_{ij},$

$v_{ij})_{j\in\varsigma_i,\, i\in[m]}$ is a g-fusion frame for H, with the optimal lower bound A. Then by the Archimedean Property, there exists an $r \in \mathbb{N}$, such that $r > \frac{2}{A}$. Now, there exists $f_r \in H$, such that $\|f_r\| = 1$ and we have

$$\begin{aligned}
&\sum_{i\in[m]}\sum_{j\in\varsigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}} f_r\|^2 \\
&\quad = \sum_{i\in[m]}\sum_{j\in\tau_{(r+1)i}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}} f_r\|^2 + \sum_{i\in[m]}\sum_{j\in\varsigma_i\cap\mathbb{J}\setminus(\mathbb{J}_1\bigcup\cdots\bigcup\mathbb{J}_r)} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}} f_r\|^2 \\
&\quad \leqslant \sum_{i\in[m]}\sum_{j\in(\tau_{ri}\bigcup\sigma_{ri})} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}} f_r\|^2 + \sum_{i\in[m]}\sum_{j\in\bigcup_{k\geqslant r+1}\mathbb{J}_k} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}} f_r\|^2 \\
&\qquad < \frac{1}{r} + \frac{1}{r} < A\|f_r\|^2,
\end{aligned}$$

and this is a contradiction with the lower bound property of A. □

Corollary 5. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J},\, i\in[m]}$ be a g-fusion woven frame for H. Then there exist a collection of disjoint finite subsets $\{\tau_i\}_{i\in[m]}$ of $\mathbb{J}$ and $A > 0$, such that for each partition $\{\sigma_i\}_{i\in[m]}$ of the set $\mathbb{J}\setminus\bigcup_{i\in[m]}\tau_i$, the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in(\sigma_i\bigcup\tau_i),\, i\in[m]}$ is a g-fusion frame for H with the lower frame bound A.*

Corollary 6. *Suppose that $(W_j.\Lambda_j, w_j)_{j\in\mathbb{J}}$ and $(V_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ are g-fusion frames for H with the optimal upper frame bounds B_1 and B_2, respectively, and they constitute a g-fusion woven frame for H. Then, $B_1 + B_2$ is not an optimal upper g-fusion woven frame bound.*

Proof. Let $\varepsilon > 0$. Assume that $B_1 + B_2$ is the optimal upper g-fusion woven frame bound for the g-fusion woven frame. So, there exists $\sigma \subset \mathbb{J}$, such that

$$\sup_{\|f\|=1}\Big(\sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j} f\|^2 + \sum_{j\in\sigma^c} v_j^2\|\Theta_j\pi_{V_j} f\|^2\Big) = B_1 + B_2.$$

Therefore, there exists $f_1 \in H$, such that $\|f_1\| = 1$ and

$$\sum_{j\in\sigma} w_j^2\|\Lambda_j\pi_{W_j} f_1\|^2 + \sum_{j\in\sigma^c} v_j^2\|\Theta_j\pi_{V_j} f_1\|^2 \geqslant B_1 + B_2 - \varepsilon.$$

Thus, by the assumption,

$$\sum_{j\in\mathbb{J}\setminus\sigma} w_j^2\|\Lambda_j\pi_{W_j} f_1\|^2 + \sum_{j\in\mathbb{J}\setminus\sigma^c} v_j^2\|\Theta_j\pi_{V_j} f_1\|^2 \leqslant \varepsilon$$

and this is a contradiction by Theorem 7. □

Theorem 8. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}$ be a g-fusion frame for H with bounds A_i and B_i for each $i \in [m]$. Suppose that there exists $N > 0$, such that for all $i, k \in [m]$ with $i \neq k$, $\mathbb{I} \subset \mathbb{J}$ and $f \in H$, the following inequality is valid:*

$$\sum_{j\in\mathbb{I}} \|(v_{ij}\Lambda_{ij}\pi_{W_{ij}} - v_{kj}\Lambda_{kj}\pi_{W_{kj}})f\|^2 \leqslant N \min\Big\{\sum_{j\in\mathbb{I}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2, \sum_{j\in\mathbb{I}} v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big\}.$$

Then the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ is woven frame with universal bounds.

$$\frac{A}{(m-1)(N+1)+1} \quad and \quad B,$$

where $A := \sum_{i\in[m]} A_i$ and $B := \sum_{i\in[m]} B_i$.

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be a partition of $\mathbb{J}$ and $f \in H$. Therefore,

$$\begin{aligned}
\sum_{i\in[m]} A_i\|f\|^2 &\leqslant \sum_{i\in[m]}\sum_{j\in\mathbb{J}} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 = \sum_{i\in[m]}\sum_{k\in[m]}\sum_{j\in\sigma_k} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \\
&\leqslant \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 \\
&\quad + \sum_{\substack{k\in[m]\\ k\neq i}}\sum_{j\in\sigma_k}\Big\{\|v_{ij}\Lambda_{ij}\pi_{W_{ij}}f - v_{kj}\Lambda_{kj}\pi_{W_{kj}}f\|^2 + v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big\}\Big) \\
&\leqslant \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2 + \sum_{\substack{k\in[m]\\ k\neq i}}\sum_{j\in\sigma_k}(N+1)v_{kj}^2\|\Lambda_{kj}\pi_{W_{kj}}f\|^2\Big) \\
&= \{(m-1)(N+1)+1\}\sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\Big).
\end{aligned}$$

Thus, by Proposition 1, we get

$$\frac{A}{(m-1)(N+1)+1}\|f\|^2 \leqslant \sum_{i\in[m]}\Big(\sum_{j\in\sigma_i} v_{ij}^2\|\Lambda_{ij}\pi_{W_{ij}}f\|^2\Big) \leqslant B\|f\|^2.$$

□

Now let us formulate the main result of this section. By Corollary 5 and Theorem 8, we conclude that the following statement holds:

Theorem 9. *A g-fusion frame $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}, i\in[m]}$ for H is woven frame if and only if it is a weakly woven frame.*

4. Weaving gf-Riesz bases. Weaving Riesz bases and some results are presented in [1]. Bemrose et al. [1] managed to do this firstly in the finite case and so, the result was extended in the Theorem 5.2 for arbitrary sets. The following is a general result:

Theorem 10. *Let $(W_j, \Lambda_j, w_j)_{j\in\mathbb{N}}$ and $(V_j, \Theta_j, v_j)_{j\in\mathbb{N}}$ be two gf-Riesz bases with common constants $0 < A \leqslant B < \infty$, such that for each $\sigma \subset \mathbb{N}$, the family $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a gf-Riesz sequence with Riesz bounds A and B. Then, $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a gf-Riesz basis.*

Proof. Let $\sigma \subset \mathbb{N}$ and assume that $n := |\sigma| < \infty$. If $n = 0$, then the proof is evident. Suppose that the result holds for each n. Let $\sigma \subset \mathbb{N}$ with $|\sigma| = n+1$ and choose $j_0 \in \sigma$. Suppose that $\sigma_1 := \sigma\backslash\{j_0\}$, so $(W_j, \Lambda_j, w_j)_{j\in\sigma_1} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma_1^c}$ is a gf-Riesz basis by the induction hypothesis. Assume that $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is not a gf-Riesz basis. For any $g \in H_{j_0}$, if

$$\pi_{V_{j_0}} \Theta_{j_0}^* g \in \operatorname{span}\big(\{\pi_{W_j} \Lambda_j^* H_j\}_{j\in\sigma} \bigcup \{\pi_{V_j} \Theta_j^* H_j\}_{j\in\sigma^c}\big),$$

then

$$\overline{\operatorname{span}}\big(\{\pi_{W_j} \Lambda_j^* H_j\}_{j\in\sigma} \bigcup \{\pi_{V_j} \Theta_j^* H_j\}_{j\in\sigma^c}\big) \supset \overline{\operatorname{span}}\big(\{\pi_{W_j} \Lambda_j^* H_j\}_{j\in\sigma_1} \bigcup \{\pi_{V_j} \Theta_j^* H_j\}_{j\in\sigma_1^c}\big) = H,$$

i.e., $\{\pi_{W_j} \Lambda_j^* H_j\}_{j\in\sigma} \bigcup \{\pi_{V_j} \Theta_j^* H_j\}_{j\in\sigma^c}$ would be gf-complete, which is assumed to not be in the case. Thus, the following must hold:

$$\pi_{V_{j_0}} \Theta_{j_0}^* g \notin \operatorname{span}\big(\{\pi_{W_j} \Lambda_j^* H_j\}_{j\in\sigma} \bigcup \{\pi_{V_j} \Theta_j^* H_j\}_{j\in\sigma^c}\big).$$

It follows that

$$(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c} \bigcup (V_{j_0}, \Theta_{j_0}, v_{j_0})$$

is a gf-Riesz sequence in H. On the other hand, $\sigma_1^c = \sigma^c \bigcup \{j_0\}$,

$$(W_j, \Lambda_j, w_j)_{j\in\sigma_1} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma_1^c}$$

cannot be a gf-Riesz basis, because it was obtained by deleting the element $(W_{j_0}, \Lambda_{j_0}, w_{j_0})$ from a gf-Riesz sequence; this leads to a contradiction.

Next, for the infinite case, suppose that there exists a $\sigma \subset \mathbb{N}$, such that $|\sigma| = |\sigma^c| = \infty$, such that

$$H_1 := \overline{\operatorname{span}}\big(\{\pi_{W_j}\Lambda_j^* H_j\}_{j\in\sigma} \bigcup \{\pi_{V_j}\Theta_j^* H_j\}_{j\in\sigma^c}\big) \neq H.$$

Let $0 \neq f \in H_1^{\perp}$. Since $(V_j, \Theta_j, v_j)_{j\in\mathbb{N}}$ is a g-fusion Bessel sequence, by taking the tail of the series, there is a $\sigma_1 \subset \sigma$ with $|\sigma_1| < \infty$ and

$$\sum_{j\in\sigma\backslash\sigma_1} v_j^2 \|\Theta_j \pi_{V_j} f\|^2 < \frac{A}{2}\|f\|^2.$$

From the first part of the proof, the following family is a gf-Riesz basis with bounds A and B:

$$(W_j, \Lambda_j, w_j)_{j\in\sigma_1} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma\backslash\sigma_1^c} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$$

Therefore,

$$\begin{aligned} A\|f\|^2 &\leqslant \sum_{j\in\sigma_1} w_j^2 \|\Lambda_j \pi_{W_j} f\|^2 + \sum_{j\in\sigma\backslash\sigma_1} v_j^2 \|\Theta_j \pi_{V_j} f\|^2 + \sum_{j\in\sigma^c} v_j^2 \|\Theta_j \pi_{V_j} f\|^2 \\ &= \sum_{j\in\sigma\backslash\sigma_1} v_j^2 \|\Theta_j \pi_{V_j} f\|^2 < \frac{A}{2}\|f\|^2 \end{aligned}$$

and it is a contradiction. □

Now, by induction on i, we can get the following result:

Corollary 7. *Let $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{N}, i\geqslant 2}$ be a countable family of gf-Riesz bases for H and there are common constants $0 < A \leqslant B < \infty$, so that for each partition $\{\sigma_i\}_{i\geqslant 2}$ of $\mathbb{N}$, the family $\bigcup_{i=2}^{\infty} (W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\sigma_i}$ be a gf-Riesz sequence with Riesz bounds A and B. Then, $(W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{N}, i\geqslant 2}$ is actually a gf-Riesz basis.*

Theorem 11. *Let $(W_j, \Lambda_j, w_j)_{j\in\mathbb{N}}$ and $(V_j, \Theta_j, v_j)_{j\in\mathbb{N}}$ be two gf-Riesz bases and let there be a common constant $0 < A$, such that for each $\sigma \subset \mathbb{N}$, the family $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a g-fusion frame with the lower bound A. Then, $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is actually a gf-Riesz basis.*

Proof. Similar to the proof of Theorem 10, let $\sigma \subset \mathbb{N}$ and assume that $|\sigma| := n < \infty$. If $n = 0$, then the proof is obvious. Suppose that the result holds for each n and let $\sigma \subset \mathbb{N}$ with $|\sigma| = n+1$, and choose $j_0 \in \sigma$. Suppose that $\sigma_1 := \sigma \backslash \{j_0\}$, so $(W_j, \Lambda_j, w_j)_{j\in\sigma_1} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma_1^c}$ is a gf-Riesz basis by the induction hypothesis. So, $(W_j, \Lambda_j, w_j)_{j\in\sigma_1} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a gf-Riesz sequence. But $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a g-fusion frame and the removal of the single vector $(V_{j_0}, \Theta_{j_0}, v_{j_0})$ yields a set that dose not longer $\overline{\text{span}}\{\pi_{V_j}\Theta_j^* H_j\}$, thus $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ must be a gf-Riesz basis. Now, let $|\sigma| = \infty$. Choose $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma$, such that $\sigma = \bigcup_{i=1}^{\infty} \sigma_i$, and $|\sigma_i| < \infty$. The family $(W_j, \Lambda_j, w_j)_{j\in\sigma_i} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma_i^c}$ is a gf-Riesz basis with the lower bound A for each $i \geqslant 1$. If $\{g_j\}_{j\in\mathbb{N}} \in \mathscr{H}_2$ and

$$\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j \pi_{V_j}\Theta_j^* g_j = 0,$$

then

$$\begin{aligned} 0 &= \|\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j \pi_{V_j}\Theta_j^* g_j\|^2 \\ &= \lim_{i\to\infty} \|\sum_{j\in\sigma_i} w_j \pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma_i^c} v_j \pi_{V_j}\Theta_j^* g_j\|^2 \geqslant \lim_{i\to\infty} A\Big(\sum_{j\in\sigma_i}\|g_j\|^2 + \sum_{j\in\sigma_i^c}\|g_j\|^2\Big). \end{aligned}$$

So, $g_j = 0$ for all $j \geqslant 1$. This means that the synthesis operator T for the family $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ with the representation space $\tilde{\mathscr{H}_2}$ is bounded, linear, surjective, and injective. Therefore, $\tilde{\mathscr{H}_2} = \mathcal{N}^{\perp}(T)$ and so by Theorem 4.2 in [20], the family is a gf-Riesz basis. □

In the following, we show that a g-fusion frame that is not a gf-Riesz basis, cannot be woven with a gf-Riesz basis, which is a general case of Theorem 5.4 in [1].

Theorem 12. *Let $\Lambda = (W_j, \Lambda_j, w_j)_{j\in\mathbb{J}}$ be a gf-Riesz basis for H and $\Theta := (V_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ be a g-fusion frame for H. If Λ and Θ are g-fusion woven, then Θ must be a gf-Riesz basis.*

Proof. Let $\varepsilon > 0$. Suppose that Θ is not a gf-Riesz basis and it may be assumed that, by the remark of Lemma 3, $\pi_{V_{j_1}}\Theta_{j_1}^* g \in \overline{\text{span}}\{\pi_{V_j}\Theta_j^* H_j\}_{j\in\mathbb{I}\backslash\{j_1\}}$ for some $g \in H_{j_1}$ and finite subset $\mathbb{I} \subset \mathbb{J}$, where $j_1 \in \mathbb{I}$. Assume that $n := \text{card}(\mathbb{I})$ and

$$0 \leqslant d(\pi_{V_{j_1}}\Theta_{j_1}^* g, \text{span}\{\pi_{V_j}\Theta_j^* H_j\}_{j\in\mathbb{I}\backslash\{j_1\}}) < \frac{\varepsilon}{v_{j_1}^2}.$$

Let

$$\widetilde{\mathcal{H}} := \left[\operatorname{span}\{\pi_{V_j}\Theta_j^* H_j\}_{j\in\mathbb{I}\setminus\{j_1\}}\right]^{\perp},$$

Then $\widetilde{\mathcal{H}}$ has codimension at most $n-1$ in H and, since

$$\dim \operatorname{span}\{\pi_{W_j}\Lambda_j^* H_j\}_{j\in\mathbb{I}} = n,$$

there exists $f \in \operatorname{span}\{\pi_{W_j}\Lambda_j^* H_j\}_{j\in\mathbb{I}} \cap \widetilde{\mathcal{H}}$. Now, if $\sigma := \mathbb{I}$, then

$$\sum_{j\in\sigma^c} w_j^2 \|\Lambda_j \pi_{W_j} f\|^2 = \sum_{j\in\sigma^c} w_j^2 |\langle \pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j} f, f\rangle| = 0,$$

and, also,

$$\sum_{j\in\sigma} v_j^2 \|\Theta_j \pi_{V_j} f\|^2 = v_{j_1}^2 \|\Theta_{j_1}\pi_{V_{j_1}} f\|^2.$$

Let $f = f_1 + f_2$, where $f_1 \in \widetilde{\mathcal{H}}$ and $f_2 \in \widetilde{\mathcal{H}}^{\perp}$. Therefore,

$$v_{j_1}^2 \|\Theta_{j_1}\pi_{V_{j_1}} f\|^2 = v_{j_1}^2 |\langle \pi_{V_{j_1}}\Theta_{j_1}^*\Theta_{j_1}\pi_{V_{j_1}} f, f_2\rangle| < \varepsilon.$$

So, these two families are not g-fusion woven and this is a contradiction. □

Let us introduce the weaving equivalent of an unconditional gf-Basis for the Hilbert space.

Theorem 13. *Let $\Lambda = (W_j, \Lambda_j, w_j)_{j\in\mathbb{J}}$ and $\Theta := (V_j, \Theta_j, v_j)_{j\in\mathbb{J}}$ be gf-Riesz sequence for H with bounds A_1, B_1, and A_2, B_2, respectively. Then the following are equivalent:*

(I) There exist $0 < B \leqslant C < \infty$, such that for each $\sigma \in \mathbb{J}$ the family $(W_j, \Lambda_j, w_j)_{j\in\sigma} \bigcup (V_j, \Theta_j, v_j)_{j\in\sigma^c}$ is a gf-Riesz sequence with bounds B and C.

(II) There exists a $A > 0$, such that for all $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ and $\sigma \in \mathbb{J}$,

$$A\Big\|\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j\Big\|^2 \leqslant \Big\|\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j \pi_{V_j}\Theta_j^* g_j\Big\|^2.$$

(III) There exists a $D > 0$, such that for all $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ and $\sigma \in \mathbb{J}$,

$$D\Big(\Big\|\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j\Big\|^2 + \Big\|\sum_{j\in\sigma} v_j \pi_{V_j}\Theta_j^* g_j\Big\|^2\Big) \leqslant \Big\|\sum_{j\in\sigma} w_j \pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j \pi_{V_j}\Theta_j^* g_j\Big\|^2.$$

(IV) There exists a $E > 0$ *satisfying for all* $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ *and* $\sigma \in \mathbb{J}$ *the following condition: if* $\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\| = 1$*, then*

$$E \leqslant \Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2.$$

Proof. The implications $(III) \Rightarrow (II)$ and $(II) \Leftrightarrow (IV)$ are clear.

$(II) \Rightarrow (III)$. Let $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ and $\sigma \in \mathbb{J}$. We get

$$\begin{aligned}\Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2 &= \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j + \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j - \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2\\ &\leqslant \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j + \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 + \Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2\\ &\leqslant \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j+\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2+\frac{1}{A}\Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j+\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2.\end{aligned}$$

Hence,

$$\frac{A}{A+1}\Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2 \leqslant \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j + \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2.$$

Similarly,

$$\frac{A}{A+1}\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 \leqslant \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j + \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2.$$

Thus, we have

$$\begin{aligned}&\frac{1}{2}\frac{A}{A+1}\Big(\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 + \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2\Big)\\ &\qquad\leqslant \frac{A}{A+1}\max\Big(\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2, \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2\Big)\\ &\qquad\leqslant \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j + \sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2.\end{aligned}$$

$(I) \Rightarrow (II)$. Assume that $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ and $\sigma \in \mathbb{J}$. We can write

$$\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 \leqslant C\sum_{j\in\sigma}\|g_j\|^2 \leqslant C\Big(\sum_{j\in\sigma}\|g_j\|^2 + \sum_{j\in\sigma^c}\|g_j\|^2\Big)$$

$$\leqslant \frac{C}{B}\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2.$$

$(III) \Rightarrow (I)$. Let $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{H}_2$ and $\sigma \in \mathbb{J}$. We have

$$\begin{aligned}\sum_{j\in\mathbb{J}}\|g_j\|^2 &= \sum_{j\in\sigma}\|g_j\|^2 + \sum_{j\in\sigma^c}\|g_j\|^2 \leqslant \frac{1}{A_1}\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 + \frac{1}{A_2}\Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2\\ &\leqslant \max\{\frac{1}{A_1},\frac{1}{A_2}\}\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j\Big\|^2 + \Big\|\sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2\\ &\leqslant \frac{1}{D}\max\{\frac{1}{A_1},\frac{1}{A_2}\}\Big\|\sum_{j\in\sigma} w_j\pi_{W_j}\Lambda_j^* g_j + \sum_{j\in\sigma^c} v_j\pi_{V_j}\Theta_j^* g_j\Big\|^2.\end{aligned}$$

The upper bound of $(W_j,\Lambda_j,w_j)_{j\in\sigma}\bigcup(V_j,\Theta_j,v_j)_{j\in\sigma^c}$ is obvious. □

5. Perturbation for g-fusion woven.

Theorem 14. *Let $\Lambda_i := (W_{ij},\Lambda_{ij},v_{ij})_{j\in\mathbb{J}}$ be a g-fusion frame for H with frame bounds A_i and B_i for each $i \in [m]$. Suppose that there exist non-negative scalers λ_i, η_i, and μ_i, such that for some fixed $n \in [m]$:*

$$A := A_n - \sum_{i\in[m]\setminus\{n\}} (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i}) > 0$$

and

$$\begin{aligned}&\Big\|\sum_{j\in\mathbb{I}}(v_{nj}\pi_{W_{nj}}\Lambda_{nj}^* - v_{ij}\pi_{W_{ij}}\Lambda_{ij}^*)f_j\Big\| \leqslant\\ &\qquad\leqslant \eta_i\Big\|\sum_{j\in\mathbb{I}} v_{nj}\pi_{W_{nj}}\Lambda_{nj}^* f_j\Big\| + \mu_i\Big\|\sum_{j\in\mathbb{I}} v_{ij}\pi_{W_{ij}}\Lambda_{ij}^* f_j\Big\| + \lambda_i\big(\sum_{j\in\mathbb{I}}\|f_j\|^2\big)^{\frac{1}{2}}\end{aligned}$$

for every finite subset $\mathbb{I} \subset \mathbb{J}$, $f_j \in H_j$ and $i \in [m]\setminus\{n\}$. Then the family $(W_{ij},\Lambda_{ij},v_{ij})_{i\in[m],j\in\mathbb{J}}$ is a g-fusion woven frame for H with universal frame bounds A and $\sum_{i\in[m]} B_i$.

Proof. By Proposition 1, the upper frame bound is clear. For the lower frame bound, assume that T_j is the synthesis operator with the g-fusion frame Λ_j for any $i \in [m]$. Since

$$\|T_i f_j\| \leqslant \sqrt{B_i}\big(\sum_{j\in\mathbb{I}}\|f_j\|^2\big)^{\frac{1}{2}},$$

for each finite subset $\mathbb{I} \subset \mathbb{J}$ and $f_j \in H_j$, then for every $i \in [m]\backslash\{n\}$ we have

$$\|(T_n - T_i)f_j\| = \|\sum_{j\in\mathbb{I}}(\upsilon_{nj}\pi_{W_{nj}}\Lambda^*_{nj} - \upsilon_{ij}\pi_{W_{ij}}\Lambda^*_{ij})f_j\|$$

$$\leqslant \eta_i\|T_nf_j\|+\mu_i\|T_if_j\|+\lambda_i\big(\sum_{j\in\mathbb{I}}\|f_j\|^2\big)^{\frac{1}{2}} \leqslant \big(\lambda_i+\eta_i\sqrt{B_n}+\mu_i\sqrt{B_i}\big)\big(\sum_{j\in\mathbb{I}}\|f_j\|^2\big)^{\frac{1}{2}}.$$

Therefore,

$$\|T_n - T_i\| \leqslant (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i}\big). \tag{7}$$

For any $i \in [m]$ and $\sigma \in \mathbb{J}$, we define

$$T_i^{(\sigma)} : (\sum_{j\in\sigma}\oplus H_j)_{\ell_2} \longrightarrow H;$$

$$T_i^{(\sigma)}\{g_j\}_{j\in\sigma} = \sum_{j\in\sigma}\upsilon_{ij}\pi_{W_{ij}}\Lambda^*_{ij}g_j.$$

It is easy to check that $T_i^{(\sigma)}$ is a bounded operator, since $\|T_i^{(\sigma)}\{g_j\}_{j\in\sigma}\| \leqslant \|T_i\{g_j\}_{j\in\sigma}\|$. Similarly, with (7), we can show that for each $i \in [m]\backslash\{n\}$:

$$\|T_n^{(\sigma)} - T_i^{(\sigma)}\| \leqslant (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i}). \tag{8}$$

For every $f \in H$ and $i \in [m]\backslash\{n\}$, we have

$$\begin{aligned}
&\big\|\big(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_i^{(\sigma)}(T_i^{(\sigma)})^*\big)f\big\| \\
&\leqslant \big\|\big(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_n^{(\sigma)}(T_i^{(\sigma)})^*\big)f\big\| + \big\|\big(T_n^{(\sigma)}(T_i^{(\sigma)})^* - T_i^{(\sigma)}(T_i^{(\sigma)})^*\big)f\big\| \\
&\leqslant \|T_n^{(\sigma)}\|\big\|\big((T_n^{(\sigma)})^* - (T_i^{(\sigma)})^*\big)f\big\| + \|T_i^{(\sigma)}\|\big\|\big(T_n^{(\sigma)} - T_i^{(\sigma)}\big)f\big\| \\
&\leqslant (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i})\|f\|.
\end{aligned}$$

Now, suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of $\mathbb{J}$ and T is the synthesis operator associated with the g-fusion Bessel sequence $(W_{ij}, \Lambda_{ij}, \upsilon_{ij})_{i\in[m],j\in\sigma_i}$. Hence, we obtain

$$\|T^*f\|^2 = |\langle f, TT^*f\rangle| = \big|\big\langle f, \sum_{i\in[m]}\sum_{j\in\sigma_i}\upsilon_{ij}\pi_{W_{ij}}\Lambda^*_{ij}\Lambda_{ij}\pi_{W_{ij}}f\big\rangle\big|$$

$$= \big|\big\langle f, \sum_{i\in[m]\backslash\{n\}}\sum_{j\in\sigma_i}\upsilon_{ij}\pi_{W_{ij}}\Lambda^*_{ij}\Lambda_{ij}\pi_{W_{ij}}f + \sum_{i\in[m]}\sum_{j\in\sigma_i}\upsilon_{nj}\pi_{W_{nj}}\Lambda^*_{nj}\Lambda_{nj}\pi_{W_{nj}}f$$

$$
\begin{aligned}
&\quad - \sum_{i\in[m]\setminus\{n\}} \sum_{j\in\sigma_i} v_{nj}\pi_{W_{nj}}\Lambda_{nj}^*\Lambda_{nj}\pi_{W_{nj}} f\big\rangle\big| \\
&= \big|\big\langle f, \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{nj}\pi_{W_{nj}}\Lambda_{nj}^*\Lambda_{nj}\pi_{W_{nj}} f - \sum_{i\in[m]\setminus\{n\}} \sum_{j\in\sigma_i} \big(v_{nj}\pi_{W_{nj}}\Lambda_{nj}^*\Lambda_{nj}\pi_{W_{nj}} \\
&\quad v_{ij}\pi_{W_{ij}}\Lambda_{ij}^*\Lambda_{ij}\pi_{W_{ij}}\big) f\big\rangle\big| \geqslant \big|\big\langle f, \sum_{i\in[m]} \sum_{j\in\sigma_i} v_{nj}\pi_{W_{nj}}\Lambda_{nj}^*\Lambda_{nj}\pi_{W_{nj}} f\big\rangle\big| \\
&\quad - \sum_{i\in[m]\setminus\{n\}} \big|\big\langle f, \sum_{j\in\sigma_i} \big(v_{nj}\pi_{W_{nj}}\Lambda_{nj}^*\Lambda_{nj}\pi_{W_{nj}} - v_{ij}\pi_{W_{ij}}\Lambda_{ij}^*\Lambda_{ij}\pi_{W_{ij}}\big) f\big\rangle\big| \\
&\quad \geqslant |\langle f, T_n T_n^* f\rangle| - \sum_{i\in[m]\setminus\{n\}} \|f\| \big\|\big(T_n^{(\sigma_i)}(T_n^{(\sigma_i)})^* - T_i^{(\sigma_i)}(T_i^{(\sigma_i)})^*\big) f\big\| \\
&\quad \geqslant A_n\|f\|^2 - \sum_{i\in[m]\setminus\{n\}} \|f\|(\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i})\|f\| \\
&= \Big(A_n - \sum_{i\in[m]\setminus\{n\}} (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i})\Big)\|f\|^2 = \mathcal{A}\|f\|^2.
\end{aligned}
$$

This completes the proof. □

The following result is obtained when the index n in Theorem 14 is not fixed with a similar proof.

Corollary 8. *For each $i \in [m]$, let $\Lambda_i := (W_{ij}, \Lambda_{ij}, v_{ij})_{j\in\mathbb{J}}$ be a g-fusion frame for H with frame bounds A_i and B_i. Suppose that there exist non-negative scalars λ_i, η_i, μ_i and $i \in [m-1]$, such that*

$$
\mathcal{A} := A_1 - \sum_{i\in[m-1]} (\lambda_i + \eta_i\sqrt{B_n} + \mu_i\sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i}) > 0
$$

and

$$
\begin{aligned}
&\Big\|\sum_{j\in\mathbb{I}}(v_{ij}\pi_{W_{ij}}\Lambda_{ij}^* - v_{(i+1)j}\pi_{W_{(i+1)j}}\Lambda_{(i+1)j}^*) f_j\Big\| \\
&\leqslant \eta_i\Big\|\sum_{j\in\mathbb{I}} v_{ij}\pi_{W_{ij}}\Lambda_{ij}^* f_j\Big\| + \mu_i\Big\|\sum_{j\in\mathbb{I}} v_{(i+1)j}\pi_{W_{(i+1)j}}\Lambda_{(i+1)j}^* f_j\Big\| + \lambda_i\Big(\sum_{j\in\mathbb{I}}\|f_j\|^2\Big)^{\frac{1}{2}}
\end{aligned}
$$

for every finite subset $\mathbb{I} \subset \mathbb{J}$, $f_j \in H_j$ and $i \in [m-1]$. Then the family $(W_{ij}, \Lambda_{ij}, v_{ij})_{i\in[m],j\in\mathbb{J}}$ is a g-fusion woven frame for H with universal frame bounds $\mathcal{A}$ and $\sum_{i\in[m]} B_i$.

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