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MEIR-KEELER CONDENSING OPERATORS AND A FAMILY OF MEASURES OF NONCOMPACTNESS IN FRÉCHET SPACES

Abstract. In this paper, we propose the concept of Meir-Keeler (MK) condensing operators with respect to a family of measures of noncompactness (FMN) in a Fréchet space, and present a generalization of the Darbo theorem. Additionally, we state the notion of an n-variable MK condensing operator regarding an FMN and extend our findings to the n-variable context. To support our main results, we demonstrate the existence of solutions for a class of systems of n-variable functional Volterra integral equations, which can generalize many standard and couple systems.

Key words: *Meir-Keeler condensing operator, family of measures of noncompactness, Fréchet space, system of functional integral equations*

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1. Introduction. The theory of differential and integral equations has become a significant branch of nonlinear analysis with wide-ranging applications in real-world problems. To date, numerous studies have explored the existence of solutions of nonlinear differential and integral equations by means of the measure of noncompactness (abbreviated MN) as a key analytical technique [3], [4], [5], [6], [7], [8], [9], [13], [19], [23]. In particular, several research articles have focused on the asymptotic behavior of continuous solutions of certain integral and differential equations on the real half-axis in [11], [12] and references therein. These investigations have yielded some significant results through the application of MN and Darbo theorem. The analyses were performed in Banach space $BC(\mathbb{R}_+)$, which includes all bounded and continuous real-valued functions on $\mathbb{R}_+$, equipped with the standard norm. In 1980, Banaś and Goebel introduced

Definition 3. [7] *Suppose that $\mathcal{M}$ is a class of subsets of a Fréchet space $\mathcal{E}$ and $\mathfrak{N}_{\mathcal{E}}$ denotes the subfamily containing all relatively compact sets. $\mathcal{M}$ is called an admissible set when* Conv $(\mathcal{I})$ *and* $\bar{\mathcal{I}} \in \mathcal{M}$ *for any* $\mathcal{I} \in \mathcal{M}$*, and* $\mathfrak{N}_{\mathcal{B}} \cap \mathcal{M} \neq \varnothing$.

Note that a Fréchet space is a locally convex space, which is complete with respect to a translation-invariant metric. Throughout this article, presume $\mathcal{E}$ is a Fréchet space and $\mathcal{M}$ is an admissible set of $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$. In addition, $\mathfrak{N}_{\mathcal{E}}$ refers to the subfamily including all relatively compact subsets of $\mathcal{E}$. For a $\mathcal{I} \subset \mathcal{E}$, $\bar{\mathcal{I}}$, and Conv $(\mathcal{I})$ are denoted as the closure of $\mathcal{I}$ and the closed convex hull of $\mathcal{I}$, respectively.

Definition 4. [20] *Presume $\mathcal{E}$ is a Fréchet space. A family of functions $\{\mu_m\}_{m\geqslant 0}$, where $\mu_m : \mathcal{M} \to [0,\infty)$, is called an FMN in $\mathcal{E}$ if the following properties are held:*

(1) The family $\ker\{\mu_m\} = \{\mathcal{I} \in \mathcal{M} : \mu_m(\mathcal{I}) = 0 \ \text{ for all } \ m \geqslant 0\}$ *is nonempty and* $\ker\{\mu_m\} \subset \mathfrak{N}_{\mathcal{E}}$;

(2) For any $\mathcal{I} \subset \mathcal{J}$, $\mu_m(\mathcal{I}) \leqslant \mu_m(\mathcal{J})$ *for all* $m \geqslant 0$;

(3) $\mu_m(\mathrm{Conv}\,(\mathcal{I})) = \mu_m(\mathcal{I})$ *for all* $m \geqslant 0$;

(4) If $\{\mathcal{I}_n\}$ *is a sequence of closed sets in* $\mathcal{M}$ *provided that* $\mathcal{I}_{n+1} \subset \mathcal{I}_n$ *for* $n = 1, 2, \cdots$ *and* $\lim_{n\to\infty} \mu_m(\mathcal{I}_n) = 0$ *for any* $m \geqslant 0$*, then the intersection* $\mathcal{I}_\infty = \bigcap_{n=1}^{\infty} \mathcal{I}_n$ *is nonempty.*

Note that $\ker\{\mu_m\}$ mentioned in Definition 4(1) is as the kernel of the FMN $\{\mu_m\}_{m\geqslant 0}$. Note that $\mathcal{I}_\infty$ in Definition 4(4) belongs to $\ker\{\mu_m\}$. Indeed, it follows from $\mu_m(\mathcal{I}_\infty) \leqslant \mu_m(\mathcal{I}_n)$ for any $m \geqslant 0$ and $n \in \mathbb{N}$ that $\mu_m(\mathcal{I}_\infty) = 0$. Therefore, we conclude that $\mathcal{I}_\infty \in \ker\{\mu_m\}$.

In the following, a famous theorem in topology, needed in the proof of the main theorem, is recalled.

Theorem 2. [1, Tychonoff Theorem] *Presume $\mathcal{B}$ is a Hausdorff locally convex linear topological space and $\mathcal{D}$ is a convex subset of $\mathcal{B}$. Also, suppose $\mathcal{H} \colon \mathcal{D} \to \mathcal{B}$ is a continuous mapping, so that $\mathcal{H}(\mathcal{D}) \subseteq \mathcal{A} \subseteq \mathcal{D}$, in which $\mathcal{A}$ is compact. Then $\mathcal{H}$ possesses at least one fixed point.*

This article presents several novel contributions that try to extend and enrich the existing literature. First, we propose a generalization of the classical MK operators by employing an FMN within the framework of Fréchet spaces. This development significantly broadens the applicability of fixed point results to a wider class of topological vector spaces, including those without norm structures. Second, we extend our results on the classical MK operators to multi-variable MK condensing operators by applying the construction of an FMN on a Fréchet finite product space. Finally, we establish new existence results for a class of finite system of an n-variable functional nonlinear Volterra integral equations defined on the space $C(\mathbb{R}_+)$. It should be mention that while several studies have examined single Volterra-type equations on unbounded domains or Banach spaces like $BC(\mathbb{R}_+)$, the analysis of systems of such equations in spaces like $C(\mathbb{R}_+)$ is notably scarce. Thus, this extension offers greater flexibility, allowing for the inclusion of more general functional equations while still maintaining solvability. The approach presented here offers a significant improvement over past studies, which often focused on simpler, more restrictive cases. Also, the potential applications of this work extend to various fields, including physics, engineering, and economics, where multi-variable systems are prevalent.

2. Single variable MK condensing operators. The concept of a MK condensing operator regarding an FMN on a Fréchet space is introduced here. We also present several fixed point theorems.

Definition 5. *Presume $\mathcal{D} \neq \varnothing$ is a subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m\geqslant 0}$ is a given FMN on $\mathcal{E}$. An operator $\mathcal{S}\colon \mathcal{D} \to \mathcal{D}$ is called a MK condensing operator with respect to $\{\mu_m\}_{m\geqslant 0}$, if, for each $\xi > 0$, there is $\gamma > 0$ provided that*

$$\xi \leqslant \mu_m(\mathcal{I}) < \xi + \gamma \Longrightarrow \mu_m(\mathcal{S}(\mathcal{I})) < \xi$$

for every subset $\mathcal{I}$ of $\mathcal{D}$ and each $m \geqslant 0$.

As Theorem 2 is applied in our main theorem, the boundedness assumption on the domain is not required.

Theorem 3. *Suppose $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m\geqslant 0}$ is a given FMN on $\mathcal{E}$. If $\mathcal{S}\colon \mathcal{D} \to \mathcal{D}$ is a continuous and MK condensing operator with respect to $\{\mu_m\}_{m\geqslant 0}$, then $\mathcal{S}$ possesses at least a fixed point.*

Proof. By induction, we construct a sequence $\{\mathcal{D}_n\}$, where $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_n = \operatorname{Conv}(\mathcal{S}\mathcal{D}_{n-1})$ for $n \geqslant 1$. If there is a non-negative integer N provided that $\mu_m(\mathcal{D}_N) = 0$ for all $m \geqslant 0$, then $\mathcal{D}_N$ is compact. Consequently, Theorem 2 induces that $\mathcal{S}$ has a fixed point. Now, presume $\mu_m(\mathcal{D}_n) \neq 0$ for all $n \geqslant 0$ and some $m \geqslant 0$. Let $\mu_m(\mathcal{D}_n) = \xi_n$ and $\gamma_n = \gamma_n(\xi_n)$.

Applying the definition of $\mathcal{D}_n$, we get

$$\xi_{n+1} = \mu_m(\mathcal{D}_{n+1}) = \mu_m(Conv(\mathcal{S}\mathcal{D}_n)) = \mu_m(\mathcal{S}\mathcal{D}_n) \leqslant \mu_m(\mathcal{D}_n) = \xi_n.$$

Thus, $\{\xi_n\}$ forms a positive non-increasing sequence of $\mathbb{R}$ and $r \geqslant 0$ exists, such that $\xi_n \to r$ when $n \to \infty$. Now, we prove $r = 0$. If $r \neq 0$, there is a $N_0 \in \mathbb{N}$ provided that $r \leqslant \xi_n < r + \gamma(r)$ for $n > N_0$, and by the definition of a MK condensing operator with respect to $\{\mu_m\}_{m\geqslant 0}$, we have $\xi_{n+1} < r$, being a contradiction. Therefore, $r = 0$ and it can be deduced that $\mu_m(\mathcal{D}_n) \to 0$ as $n \to \infty$ for any $m \geqslant 0$. Further, as $\mathcal{D}_{n+1} \subset \mathcal{D}_n$ and using 4(4), we deduce $\mathcal{D}_\infty = \bigcap_{n=1}^{\infty} \mathcal{D}_n$ to be a nonempty, closed, and convex set and $\mathcal{D}_\infty \subset \mathcal{D}$. Furthermore, $\mathcal{D}_\infty \in \ker\{\mu_m\}$ is invariant under $\mathcal{S}$. By Theorem 2, $\mathcal{S}$ possesses a fixed point. □

Definition 6. [17] *An $\mathcal{L}$-function is a function $\chi\colon [0,\infty] \to [0,\infty]$ that satisfies the following conditions:*

1) $\chi(0) = 0$;
2) $\chi(a) > 0$ *for* $a \in (0,\infty)$;
3) *For any* $a \in (0,\infty)$, *there is a* $\gamma > 0$, *provided that* $\chi(b) \leqslant a$ *for all* $b \in [a, a+\gamma]$.

These properties define the class of $\mathcal{L}$-functions. Below are some examples of such functions.

Example 1. [2] Take the function $\chi\colon [0,\infty) \to [0,\infty)$ by $\chi(0) = 0$ and for $\mathsf{x} > 0$, $0 < \chi(\mathsf{x}) < \mathsf{x}$, which is continuous from the right. It is straightforward to verify that χ satisfies the properties of an $\mathcal{L}$-function. For instances, take $\chi(b) = kb$ in which $0 \leqslant k < 1$.

Theorem 4. *Presume $\mathcal{D} \neq \varnothing$ is a subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m\geqslant 0}$ is a given FMN on $\mathcal{E}$. Then $\mathcal{S}$ is classified as a MK condensing operator with respect to $\{\mu_m\}_{m\geqslant 0}$ iff there is an $\mathcal{L}$-function χ, such that*

$$\mu_m(\mathcal{S}\mathcal{I}) < \chi(\mu_m(\mathcal{I})) \tag{1}$$

for each $\mathcal{I} \subset \mathcal{D}$ with $\mu_m(\mathcal{I}) \neq 0$ for $m \geqslant 0$.

Proof. Sufficiency: Assume $\xi > 0$ is given. As χ is an $\mathcal{L}$-function, there is $\gamma > 0$ provided that $\chi(b) \leqslant \xi$ for $\xi \leqslant b < \xi + \gamma$. Now, consider a subset $\mathcal{I}$ of $\mathcal{D}$ provided that $\xi \leqslant \mu_m(\mathcal{I}) < \xi + \gamma(\xi)$ for each $m \geqslant 0$. Using (1), we get $\mu_m(\mathcal{SI}) < \chi(\mu_m(\mathcal{I})) \leqslant \xi$ for $m \geqslant 0$. This shows that $\mathcal{S}$ is a MK condensing operator in respect of $\{\mu_m\}_{m\geqslant 0}$.

Necessity: Presume $\mathcal{S}$ is a MK condensing operator in respect of $\{\mu_m\}_{m\geqslant 0}$. Then we are able to introduce a function $\nu\colon (0,\infty) \to (0,\infty)$ provided that

$$\xi \leqslant \mu_m(\mathcal{I}) < \xi + 2\nu(\xi) \Longrightarrow \mu_m(\mathcal{S}(\mathcal{I})) < \xi$$

for all $\xi \in (0,\infty)$ and $m \geqslant 0$. By ν, we can consider a nondecreasing function $\rho\colon (0,\infty) \to (0,\infty)$ by $\rho(b) = \inf\{\xi\colon b \leqslant \xi + \nu(\xi)\}$ for $b \in (0,\infty)$. It follows from $b \leqslant \xi + \nu(\xi)$ that $\rho(b) \leqslant b$ for all $b \in (0,\infty)$. Now, take $\psi_1\colon [0,\infty) \to [0,\infty)$ by

$$\psi_1(b) = \begin{cases} 0, & b = 0, \\ \rho(b), & b > 0 \text{ and } \min\{\xi > 0\colon b \leqslant \xi + \nu(\xi)\} \text{ exists}, \\ \dfrac{\rho(b)+b}{2}, & \text{otherwise.} \end{cases}$$

It is evident that $\psi_1(0) = 0$ and $0 < \psi_1(a) \leqslant a$ for $a \in (0,\infty)$. Now, fix $a \in (0,\infty)$. If $\psi_1(b) \leqslant a$ for each $b \in (a, a+\nu(a)]$, then we are able to choose $\gamma = \nu(a)$; otherwise, there is $\sigma \in (a, a+\nu(a)]$ provided that $\psi_1(\sigma) > a$. Since $\sigma \leqslant a + \nu(a)$, we conclude that $\rho(\sigma) \leqslant a$. When $\rho(\sigma) = a$, we get $\psi_1(\sigma) = \rho(\sigma) = a = \psi_1(\sigma)$, which is a contradiction. Therefore,

$$\rho(\sigma) < a < \psi_1(\sigma) = \frac{\rho(\sigma)+\sigma}{2}.$$

Now, select $u \in (\rho(\sigma), a)$ so that $\sigma \leqslant u + \nu(u)$ and $\gamma = a - u > 0$, and fix $b \in [a, a+\gamma]$. Since

$$b \leqslant a + \gamma = 2a - u < 2\frac{\rho(\sigma)+\sigma}{2} - \rho(\sigma) = \sigma \leqslant u + \nu(u),$$

we get $\rho(b) \leqslant u$. In conclusion,

$$\psi_1(b) \leqslant \frac{\rho(b)+b}{2} \leqslant \frac{u+a+\gamma}{2} = a.$$

Thus, ψ_1 is an $\mathcal{L}$-function. Now, pick $\mathcal{I} \in \mathcal{M}$ so that $\mu_m(\mathcal{I}) \neq 0$ for each $m \geqslant 0$. From the definition of ψ_1, there is $\xi \in (0, \psi_1(t))$ provided that $b \leqslant \xi + \nu(\xi)$ for every $b \in (0,\infty)$. Hence, there is $\xi \in (0, \psi_1(\mu_m(\mathcal{I})))$

provided that $\mu_m(\mathcal{I}) \leqslant \xi + \nu(\xi)$ for each $m \geqslant 0$. Therefore, $\mu_m(\mathcal{SI}) < < \xi \leqslant \psi_1(\mu_m(\mathcal{I}))$ and the proof ends. □

Corollary 1. *Presume $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\mathcal{S}\colon \mathcal{D} \to \mathcal{D}$ is a continuous operator satisfying $\mu_m(\mathcal{SI}) < \chi(\mu_m(\mathcal{I}))$ for all $\mathcal{I} \subset \mathcal{D}$ and $m \geqslant 0$, where $\{\mu_m\}_{m\geqslant 0}$ is a given FMN and χ is an $\mathcal{L}$-function. Then $\mathcal{S}$ possesses at least a fixed point.*

Definition 7. [2, Definition 2.8] $\kappa\colon [0,\infty) \to [0,\infty)$ *is said to be a strictly $\mathcal{L}$-function if*

1) $\kappa(0) = 0$;
2) $\kappa(a) > 0$ *for* $a > 0$;
3) *for each* $a > 0$, *there exists* $\gamma(a) > 0$, *such that* $\kappa(b) < a$ *for any* $b \in [a, a + \gamma(a)]$.

Example 2. [2, Example 2.11] Let $\kappa\colon [0,\infty) \to [0,\infty)$ be the function defined by $\kappa(t) = \ln(1+t)$. Then $\kappa(0) = \ln 1 = 0$ and $\kappa(a) = \ln(1+a) > 0$ for $a > 0$. Moreover, for each $a > 0$, set $\gamma(a) = e^a - 1 - a > 0$. If $b \in [a, a + \gamma(a)]$, then $b < e^a - 1$, so $\kappa(b) = \ln(1 + b) < a$. Thus, κ is a strictly $\mathcal{L}$-function.

Corollary 2. *Presume $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\mathcal{S}\colon \mathcal{D} \to \mathcal{D}$ is a continuous operator provided that $\mu_m(\mathcal{SI}) \leqslant \kappa(\mu_m(\mathcal{I}))$ for any $\mathcal{I} \subset \mathcal{D}$ and $m \geqslant 0$ in which $\{\mu_m\}_{m\geqslant 0}$ is a given FMN and κ is a strictly $\mathcal{L}$-function. Then $\mathcal{S}$ possesses at least one fixed point.*

Proof. It is sufficient to demonstrate that $\mathcal{S}$ is a MK condensing operator regarding $\{\mu_m\}_{m\geqslant 0}$. Let $\xi > 0$ be arbitrary. As κ is a strictly $\mathcal{L}$-function, there is a $\gamma > 0$ provided that $\kappa(b) < \xi$ for $\xi \leqslant b < \xi + \gamma$. Now, suppose $\mathcal{I}$ is a subset of $\mathcal{D}$ so that $\xi \leqslant \mu_m(\mathcal{I}) < \xi + \gamma$ for each $m \geqslant 0$. Then we can conclude that $\mu_m(\mathcal{S}(\mathcal{I})) \leqslant \kappa(\mu_m(\mathcal{I})) < \xi$. Now, applying Theorem 3, the proof ends. □

3. Multi-variable MK condensing operators. In the recent decade, many researchers have discussed the equivalence of the existence and uniqueness of n-tuples fixed points and usual fixed points for multidimensional mappings in [15], [16], [21], [22] and references therein. Following their ideas, we present some useful theorems regarding the construction of an FMN on a finite product space.

Theorem 5. [10] *Let $\{\mu_m^1\}_{m\geqslant 0},\dots,\{\mu_m^n\}_{m\geqslant 0}$ be families of measures of noncompactness on Fréchet spaces $\mathcal{E}_1,\dots,\mathcal{E}_n$, respectively. Additionally, presume $\mathcal{H}\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is a convex function fulfilling $\mathcal{H}(q_1,\dots,q_n) = 0$ iff $q_i = 0$ for each $i = 1,\dots,n$. Then $\mu_m(\mathcal{I}) = \mathcal{H}(\mu_m^1(\mathcal{I}_1),\dots,\mu_m^n(\mathcal{I}_n))$ defines an FMN on $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ for $m \geqslant 0$, where $\mathcal{I}_i$ represents the natural projections (in short, NPs) of $\mathcal{I}$ into $\mathcal{E}_i$ for each $i = 1,\dots,n$.*

As a conclusion of Theorem 5, we present the next example.

Example 3. Presume $\{\mu_m\}_{m\geqslant 0}$ is an FMN on a Fréchet space $\mathcal{E}$. Then, by considering

$$\begin{cases} \mathcal{H}_1(q_1,\dots,q_n) = \max\{q_1,\dots,q_n\},\\ \mathcal{H}_2(q_1,\dots,q_n) = \sum\limits_{i=1}^{n} q_i \end{cases}$$

for $(q_1,\dots,q_n) \in \mathbb{R}_+^n$, the conditions of Theorem 5 are held [3]. Now, presume $\mathcal{M}$ is an admissible set in Fréchet space $\mathcal{E}$ and define $\mathcal{M}_{\mathcal{E}^n} = \{\mathcal{I} \subset \mathcal{E}^n | \mathcal{I}_i \in \mathcal{M}\}$, where $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ for $i = 1,\dots,n$. It is evident that the conditions in Definition 3 are fulfilled. Therefore, the mappings $\tilde{\mu}_m^1\colon \mathcal{M}_{\mathcal{E}^n} \to [0,\infty)$ and $\tilde{\mu}_m^2\colon \mathcal{M}_{\mathcal{E}^n} \to [0,\infty)$ are defined as follows:

$$\begin{cases} \tilde{\mu}_m^1(\mathcal{I}) = \max\big\{\mu_m(\mathcal{I}_1),\dots,\mu_m(\mathcal{I}_n)\big\},\\ \tilde{\mu}_m^2(\mathcal{I}) = \sum\limits_{i=1}^{n} \mu_m(\mathcal{I}_i), \end{cases}$$

which are two families of measures of noncompactness in the space $\mathcal{E}^n$ for $m \geqslant 0$ and $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ for $i = 1,\dots,n$.

Next, we introduce an n-variable MK condensing operator regarding the family $\{\mu_m\}_{m\geqslant 0}$. Let $\mathcal{D}^n$ represent the finite product of $\mathcal{D}$ and $\mathcal{M}$ be an admissible set in Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$.

Definition 8. *Presume $\mathcal{D}$ is a nonempty subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m\geqslant 0}$ is a given FMN on $\mathcal{E}$. An operator $\mathcal{S}\colon \mathcal{D}^n \to \mathcal{D}$ is called a MK condensing operator with respect to $\{\mu_m\}_{m\geqslant 0}$ if, for any $\xi > 0$, there is $\gamma > 0$ provided that*

$$\xi \leqslant \max\big\{\mu_m(\mathcal{I}_1),\dots,\mu_m(\mathcal{I}_n)\big\} < \xi + \gamma \Longrightarrow \mu_m(\mathcal{S}(\mathcal{I})) < \xi$$

for each subset $\mathcal{I}$ of $\mathcal{D}^n$ and each $m \geqslant 0$, where $\mathcal{I}_i$ is the NPs of $\mathcal{I}$ into $\mathcal{D}$ for $i = 1,\dots,n$.

Theorem 6. *Presume $\mathcal{D} \neq \emptyset$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m \geqslant 0}$ is a given FMN on $\mathcal{E}$. If $\mathcal{S}_i \colon \mathcal{D}^n \to \mathcal{D}$ is a continuous and MK condensing operator with respect to $\{\mu_m\}_{m \geqslant 0}$ for $i = 1, \ldots, n$, then there is $(q_1, \ldots, q_n) \in \mathcal{D}^n$ provided that $\mathcal{S}_i(q_1, \ldots, q_n) = q_i$ for $i = 1, \ldots, n$.*

Proof. First, by Example 3, an FMN on $\mathcal{E}^n$ can be defined as follows:

$$\tilde{\mu}_m(\mathcal{I}) = \max\left\{\mu_m(\mathcal{I}_1), \mu_m(\mathcal{I}_2), \ldots, \mu_m(\mathcal{I}_n)\right\} \tag{2}$$

for any subset $\mathcal{I}$ in an admissible set $\mathcal{M}$ of $\mathcal{E}^n$ and each $m \geqslant 0$, where $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ for $i = 1, \ldots, n$. Additionally, it is clear that $\tilde{\mathcal{S}} \colon \mathcal{D}^n \to \mathcal{D}$ defined by

$$\tilde{\mathcal{S}}(q_1, \ldots, q_n) = (\mathcal{S}_1(q_1, \ldots, q_n), \mathcal{S}_2(q_1, \ldots, q_n), \ldots, \mathcal{S}_n(q_1, \ldots, q_n))$$

is continuous on $\mathcal{D}^n$. We now claim that $\tilde{\mathcal{S}}$ fulfills all hypotheses of Theorem 3. To show this, suppose that $\xi > 0$ and $\gamma(\xi) > 0$ are as in Definition 8. If $\mathcal{I}$ is a subset of $\mathcal{D}^n$ so that $\xi \leqslant \tilde{\mu}_m(\mathcal{I}) < \xi + \gamma(\xi)$ for $m \geqslant 0$, then

$$\xi < \max\left\{\mu_m(\mathcal{I}_1), \ldots, \mu_m(\mathcal{I}_n)\right\} < \xi + \gamma(\xi)$$

in which $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ for $i = 1, \ldots, n$. Using Definition 4(2) and (2), we get

$$\begin{aligned}\tilde{\mu}_m(\tilde{\mathcal{S}}(\mathcal{I})) &\leqslant \tilde{\mu}_m(\mathcal{S}_1(\mathcal{I}) \times \cdots \times \mathcal{S}_n(\mathcal{I})) \\ &= \max\left\{\mu_m(\mathcal{S}_1(\mathcal{I})), \ldots, \mu_m(\mathcal{S}_n(\mathcal{I}))\right\} < \xi.\end{aligned}$$

Now, it follows from Theorem 3 that $\tilde{\mathcal{S}}$ possesses at least a fixed point in $\mathcal{D}^n$. Therefore, there is $(q_1, \ldots, q_n) \in \mathcal{D}^n$ provided that

$$\tilde{\mathcal{S}}(q_1, \ldots, q_n) = (q_1, \ldots, q_n) = (\mathcal{S}_1(q_1, \ldots, q_n), \ldots, \mathcal{S}_n(q_1, \ldots, q_n)).$$

The proof is completed. □

Definition 9. [13] *$(q, p) \in \mathcal{I} \times \mathcal{I}$ is called a coupled fixed point of an operator $\mathcal{H} : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ when $\mathcal{H}(q, p) = q$ and $\mathcal{H}(p, q) = p$.*

Theorem 7. *Presume $\mathcal{D} \neq \emptyset$ is a bounded, closed, and convex subset of a Fréchet space $\mathcal{E}$ and $\{\mu_m\}_{m \geqslant 0}$ is a given FMN on $\mathcal{E}$. Suppose that $\mathcal{S} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ is a continuous and MK condensing operator with respect to $\{\mu_m\}_{m \geqslant 0}$. Then $\mathcal{S}$ possesses a coupled fixed point.*

Proof. Consider $\mathcal{S}_i : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ for $i = 1, 2$ as follows:

$$\begin{cases} \mathcal{S}_1(q,p) = \mathcal{S}(q,p), \\ \mathcal{S}_2(q,p) = \mathcal{S}(p,q). \end{cases}$$

The proof is the conclusion of Theorem 6. □

Next, we prove several fixed-point results using strictly $\mathcal{L}$-functions.

Theorem 8. *Presume $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\mathcal{S}_i : \mathcal{D}^n \to \mathcal{D}$ are continuous operators, provided that*

$$\mu_m(\mathcal{S}_i(\mathcal{I}_1 \times \cdots \times \mathcal{I}_n)) \leqslant \kappa\Big(\max\big\{\mu_m(\mathcal{I}_1), \ldots, \mu_m(\mathcal{I}_n)\big\}\Big) \tag{3}$$

for every bounded subset $\mathcal{I}$ of $\mathcal{D}^n$ and each $m \geqslant 0$, where $\{\mu_m\}_{m \geqslant 0}$ is an arbitrary FMN on $\mathcal{E}$, $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ into $\mathcal{D}$ for $i = 1, \ldots, n$, and κ is a strictly $\mathcal{L}$-function. Then there is $(q_1, \ldots, q_n) \in \mathcal{D}^n$, so that $\mathcal{S}_i(q_1, \ldots, q_n) = q_i$ for $i = 1, \ldots, n$.

Proof. Take a mapping $\tilde{\mathcal{S}} : \mathcal{D}^n \to \mathcal{D}^n$ by

$$\tilde{\mathcal{S}}(q_1, \ldots, q_n) = (\mathcal{S}_1(q_1, \ldots, q_n), \mathcal{S}_2(q_1, \ldots, q_n), \ldots, \mathcal{S}_n(q_1, \ldots, q_n)),$$

which is continuous. By Example 3, $\tilde{\mu}_m(\mathcal{I}) = \max\big\{\mu_m(\mathcal{I}_1), \ldots, \mu_m(\mathcal{I}_n)\big\}$ defines an FMN on $\mathcal{E}^n$ for any subset $\mathcal{I}$ in an admissible set $\mathcal{M}_{\mathcal{E}^n}$ of $\mathcal{E}^n$ and each $m \geqslant 0$ in which $\mathcal{I}_i$ is the NPs of $\mathcal{I}$ for $i = 1, \ldots, n$. Using (3), we have

$$\begin{aligned} \tilde{\mu}_m(\tilde{\mathcal{S}}(\mathcal{I})) &\leqslant \tilde{\mu}_m(\mathcal{S}_1(\mathcal{I}) \times \mathcal{S}_2(\mathcal{I}) \times \cdots \times \mathcal{S}_n(\mathcal{I})) \\ &= \max\big\{\mu_m(\mathcal{S}_1(\mathcal{I})), \mu_m(\mathcal{S}_2(\mathcal{I})), \ldots, \mu_m(\mathcal{S}_n(\mathcal{I}))\big\} \\ &\leqslant \kappa\Big(\max\big\{\mu_m(\mathcal{I}_1), \mu_m(\mathcal{I}_2), \ldots, \mu_m(\mathcal{I}_n)\big\}\Big) \\ &= \kappa(\tilde{\mu}_m(\mathcal{I})). \end{aligned}$$

Now, the proof is followed by Corollary 2. □

Theorem 9. *Presume $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\mathcal{S}_i : \mathcal{D}^n \to \mathcal{D}$ are continuous operators, provided that*

$$\mu_m(\mathcal{S}_i(\mathcal{I}_1 \times \cdots \times \mathcal{I}_n)) \leqslant \frac{1}{n}\kappa\Big(\sum_{i=1}^{n} \mu_m(\mathcal{I}_i)\Big)$$

for any bounded subset $\mathcal{I}$ of $\mathcal{D}^n$ and each $m \geqslant 0$, where $\{\mu_m\}_{m \geqslant 0}$ is a given FMN on $\mathcal{E}$, $\mathcal{I}_i$ represents the NPs of $\mathcal{I}$ into $\mathcal{D}$ for $i = 1, \ldots, n$, *and κ is a strictly $\mathcal{L}$-function. Then there is $(q_1, \ldots, q_n) \in \mathcal{D}^n$, such that $\mathcal{S}_i(q_1, \ldots, q_n) = q_i$ for $i = 1, \ldots, n$.*

Proof. From Example 3, take an FMN on $\mathcal{E}^n$ by $\tilde{\mu}_m(\mathcal{I}) = \sum_{i=1}^{n} \mu_m(\mathcal{I}_i)$ for each subset $\mathcal{I}$ in an admissible set $\mathcal{M}_{\mathcal{E}^n}$ of $\mathcal{E}^n$ and all $m \geqslant 0$, where $\mathcal{I}_i$ is the NPs of $\mathcal{I}$ for $i = 1, \ldots, n$. The proof is the same as the proof of the Theorem 8. □

Theorem 10. *Presume $\mathcal{D} \neq \varnothing$ is a closed and convex subset of a Fréchet space $\mathcal{E}$ with $\mathcal{D} \in \mathcal{M}$ and $\{\mu_m\}_{m \geqslant 0}$ is a given FMN on $\mathcal{E}$. If $\mathcal{S}_i \colon \mathcal{D}^i \to \mathcal{D}$ is a continuous and MK condensing operator with respect to $\{\mu_m\}_{m \geqslant 0}$ for $i = 1, \ldots, n$, then there is $(q_1, q_2, \ldots, q_i) \in \mathcal{D}^i$, such that*

$$\mathcal{S}_1(q_1) = q_1, \mathcal{S}_2(q_1, q_2) = q_2, \quad \ldots, \quad \mathcal{S}_i(q_1, \ldots, q_i) = q_i$$

for $i = 1, \ldots, n$.

Proof. Define $\tilde{\mathcal{S}} \colon \mathcal{D}^n \to \mathcal{D}^n$ by

$$\tilde{\mathcal{S}}(q_1, \ldots, q_n) = (\mathcal{S}_1(q_1), \mathcal{S}_2(q_1, q_2), \ldots, \mathcal{S}_n(q_1, \ldots, q_n)).$$

It is clear that $\tilde{\mathcal{S}}$ is continuous. The proof similarly comes after the proof of Theorem 6. □

4. Applications. Up until now, many applications related to showing the existence of solution of function Volterra integral equations have been presented by many researchers [3], [6], [7], [19], [20], [23], [24], [25]; however, we demonstrate the existence of solutions for a class of systems of n-variable functional Volterra integral equations, which are more useful than previous problems. In fact, since we consider $C(\mathbb{R}_+)$ and apply n-variable system instead of one variable or coupled systems with weaker conditions, this section can cover former applications. Note that Volterra integral equations, particularly those with multiple variables, play a significant role in modeling complex phenomena in various scientific fields. We first gather some basic definitions and facts that will be needed. Presume $C(\mathbb{R}_+) = \{q \colon \mathbb{R}_+ \to \mathbb{R}, q \text{ is continuous}\}$ is furnished with the family of seminorms $|q|_m = \sup\{|q(b)| \colon b \in [0, m]\}$ for $m \geqslant 0$. Then $C(\mathbb{R}_+)$ is a Fréchet space when furnished with the metric

$$d(q, p) = \sup\left\{\frac{1}{2^m} \min\{1, |q - p|_m\} \colon m \geqslant 0\right\}.$$

$\varnothing \neq \mathcal{I} \subset C(\mathbb{R}_+)$ is called bounded when $\sup\{|q|_m \colon q \in \mathcal{I}\} < \infty$ for $m \geqslant 0$. Also, let us now recall three key facts that are necessary:

1) A sequence (q_n) converges to $q \in C(\mathbb{R}_+)$ iff (q_n) is uniformly convergent to q on a compact subset of $\mathbb{R}_+$.
2) A family $\mathcal{A} \subset C(\mathbb{R}_+)$ is relatively compact iff, for any $m > 0$, the restriction to $[0, m]$ of each function from $\mathcal{A}$ constructs an equicontinuous and uniformly bounded set.
3) An operator $\mathcal{H}\colon (C(\mathbb{R}_+))^n \to C(\mathbb{R}_+)$ is continuous on $(C(\mathbb{R}_+))^n$ iff $\mathcal{H}|_m\colon (C[0, m])^n \to C[0, m]$ is continuous for all $m \geqslant 0$, where

$$\mathcal{H}|_m(q_1, \dots, q_n)(b) = \mathcal{H}(q_1, \dots, q_n)(b)$$

for $b \in [0, m]$.

Moreover, let us define an FMN in $C(\mathbb{R}_+)$ by

$$\mathcal{M}_{C(\mathbb{R}_+)} = \{\mathcal{I} \subset C(\mathbb{R}_+)\colon \mathcal{I}|_m \text{ is bounded for all } m \geqslant 0\},$$

where $\mathcal{I}|_m$ is the restriction of all functions from $\mathcal{I}$ to $[0, m]$. It is obvious that all the conditions of Definition 3 are satisfied. Fix a positive number $m > 0$ and a nonempty set $\mathcal{I}$ of $\mathcal{M}_{C(\mathbb{R}_+)}$, which is an admissible set of $C(\mathbb{R}_+)$. For $q \in \mathcal{I}$ and $\xi > 0$, the modulus of continuity of q on $[0, m]$, denoted by $w^m(q, \xi)$, is

$$w^m(q, \xi) = \sup\{|q(b) - q(a)| : a, b \in [0, m], |b - a| \leqslant \xi\}.$$

Additionally, take

$$w^m(\mathcal{I}, \xi) = \sup\{w^m(q, \xi) : q \in \mathcal{I}\}, \quad w_0^m(\mathcal{I}) = \lim_{\xi \to 0} w^m(\mathcal{I}, \xi).$$

Note that $\{w_0^m\}_{m \in \mathbb{N}}$ forms an FMN in $C(\mathbb{R}_+)$ [20]. Now, consider

$$q_i = f_i\Big(q_1(b), \dots, q_n(b), \int_0^b g(b, a, q_1(a), \dots, q_n(a))da\Big) \tag{4}$$

with the following conditions:

(i) $f_i\colon \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous functions satisfying

$$|f_i(q_1, \dots, q_n, u) - f_i(p_1, \dots, p_n, v)| \leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i - p_i|\}\Big) + |u - v|$$

in which $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing strictly $\mathcal{L}$-function with $\kappa(b) \leqslant b$, and $M = \sup_{i=1,\dots,n} f_i(0, \dots, 0)$.

(ii) The function $g\colon \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and for $b \in [0, m]$ and $q_1, \dots, q_n \in C(\mathbb{R}_+)$, there exists positive sequence constant D_m for $m \geqslant 0$ that

$$\int_0^b |g(b, a, q_1(a), \dots, q_n(a))| da < D_m$$

(iii) There is a positive sequence solution of $\kappa(r_m) + M + D_m \leqslant r_m$ for $m \geqslant 0$.

Theorem 11. *By conditions (i)–(iii), (4) has at least one solution in* $(C(\mathbb{R}_+))^n$.

Proof. First take $\mathcal{H}_i\colon (C(\mathbb{R}_+))^n \to C(\mathbb{R}_+)$ for $i \in \{1, \dots, n\}$ by

$$\mathcal{H}_i(q_1, \dots, q_n)(b) = f_i\Big(q_1(b), \dots, q_n(b), \int_0^b g(b, a, q_1(a), \dots, q_n(a)) da\Big)$$

for any $b \in \mathbb{R}_+$. It is obvious that $\mathcal{H}_i(q_1, \dots, q_n)$ is continuous on $\mathbb{R}_+$ for each $(q_1, \dots, q_n) \in (C(\mathbb{R}_+))^n$. In the following steps, we exam the validity of all hypotheses of the Theorem 8.

Step 1. We show $\mathcal{H}_i|_m\colon (C[0, m])^n \to C[0, m]$ is continuous for $m \geqslant 0$ and $i \in \{1, \dots, n\}$. For this, take $\xi > 0$ and fix $q_i \in C[0, m]$ for all $p_i \in C[0, m]$, so that $|q_i - p_i|_m \leqslant \xi$ for any $i \in \{1, \dots, n\}$. For $b \in [0, m]$, we have

$$\begin{aligned}
&\big|\mathcal{H}_i(q_1, \dots, q_n)(b) - \mathcal{H}_i(p_1, \dots, p_n)(b)\big| \\
&\leqslant \kappa\Big(\max_{i=1,\dots,n} \{|q_i(b) - p_i(b)|\}\Big) \\
&\quad + \bigg|\int_0^b g(b, a, q_1(a), \dots, q_n(a)) da - \int_0^b g(b, a, p_1(a), \dots, p_n(a)) da\bigg| \\
&\leqslant \kappa\Big(\max_{i=1,\dots,n} \{|q_i(b) - p_i(b)|\}\Big) \\
&\quad + \int_0^b \big|g(b, a, q_1(a), \dots, q_n(a)) - g(b, a, p_1(a), \dots, p_n(a))\big| da \\
&\leqslant \kappa\Big(\max_{i=1,\dots,n} \{|q_i - p_i|_m\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\int\limits_0^m \big|g(b,a,q_1(a),\dots,q_n(a)) - g(b,a,p_1(a),\dots,p_n(a))\big|da \\
&\leqslant \kappa(\xi) + mV(\xi) \leqslant \xi + mV(\xi),
\end{aligned}
$$

where

$$
\begin{aligned}
&\sigma = \sup\{|q_i(b)| \colon b \in [0,m], 1 \leqslant i \leqslant n\},\\
&V(\xi) = \sup\big\{|g(b,a,q_1(a),\dots,q_n(a)) - g(b,a,p_1(a),\dots,p_n(a))| \colon \\
&\qquad\qquad q_i, p_i \in [-\sigma-\xi, \sigma+\xi], |q_i - p_i| \leqslant \xi,\, a,b \in [0,m]\big\}.
\end{aligned}
$$

It follows from the uniform continuity of g on $[0,m]^2 \times [-\sigma-\xi, \sigma+\xi]^n$ that $V(\xi) \to 0$ as $\xi \to 0$. Thus, the first step of the proof has been completed.

Step 2. Let $b \in [0,m]$ and $q = (q_1,\dots,q_n) \in (C(\mathbb{R}_+))^n$. Then

$$
\begin{aligned}
|\mathcal{H}_i q(b)| &= \Big|f_i(q_1(b),\dots,q_n(b), \int\limits_0^b g(b,a,q_1(a),\dots,q_n(a))da)\Big| \\
&\leqslant \Big|f_i\Big(q_1(b),\dots,q_n(b), \int\limits_0^b g(b,a,q_1(a),\dots,q_n(a))da\Big) - f_i(0,\dots,0)\Big| \\
&\qquad\qquad + |f_i(0,\dots,0)| \\
&\leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i(b)|\}\Big) + \int\limits_0^b |g(b,a,q_1(a),\dots,q_n(a))da| + |f_i(0,\dots,0)| \\
&\qquad\qquad \leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i|_m\}\Big) + M + D_m.
\end{aligned}
$$

Thus,

$$
|\mathcal{H}_i q|_m \leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i|_m\}\Big) + M + D_m. \tag{5}
$$

Define $\mathcal{D} = \{q_i \in C(\mathbb{R}_+) |q_i|_m < r_m, m \geqslant 0\}$. It follows from (5) that $\mathcal{H}_i$ transforms $\mathcal{D}^n$ into $\mathcal{D}$.

Step 3. Now, fix a positive number $m > 0$ and assume $b_1, b_2 \in [0,m]$, $|b_1 - b_2| \leqslant \xi$ and $q = (q_1,\dots,q_n) \in \mathcal{D}^n$. Then

$$
|\mathcal{H}_i q(b_1) - \mathcal{H}_i q(b_2)|
$$

$$= \Big| f_i\Big(q_1(b_1), \dots, q_n(b_1), \int\limits_0^{b_1} g(b_1, a, q_1(a), \dots, q_n(a))da\Big)$$

$$- f_i\Big(q_1(b_2), \dots, q_n(b_2), \int\limits_0^{b_2} g(b_2, a, q_1(a), \dots, q_n(a))da\Big)\Big|$$

$$\leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i(b_1) - q_i(b_2)|\}\Big)$$

$$+ \int\limits_0^{b_1} |g(b_1, a, q_1(a), \dots, q_n(a)) - g(b_2, a, q_1(a), \dots, q_n(a))|da$$

$$+ \int\limits_{b_1}^{b_2} |g(b_2, a, q_1(a), \dots, q_n(a))|da$$

$$\leqslant \kappa\Big(\max_{i=1,\dots,n}\{|q_i(b_1) - q_i(b_2)|\}\Big) + mV^m(\xi) + |b_1 - b_2|U^m$$

$$\leqslant \kappa\Big(\max_{i=1,\dots,n}\{w^m(\mathcal{I}_i, \xi)\}\Big) + mV^m(\xi) + \xi U^m,$$

where

$$V^m(\xi) = \sup\big\{|g(b_1, a, z_1, \dots, z_n) - g(b_2, a, z_1, \dots, z_n)| : |b_1 - b_2| \leqslant \xi, \quad a, b_1, b_2 \in [0, m]\big\},$$

$$U^m = \sup\big\{|g(b, a, z_1, \dots, z_n)| : a, b \in [0, m], |z_i| \leqslant r_m\big\}.$$

On the other hand, as g is uniformly continuous on $[0, m]^2 \times [-\xi, +\xi]^n$, we have $V^m(\xi) \to 0$ as $\xi \to 0$. Therefore,

$$w^m(\mathcal{H}_i(\mathcal{I}_1 \times \dots \times \mathcal{I}_n, \xi)) \leqslant \kappa(\max_{i=1,\dots,n}\{w^m(\mathcal{I}_i, \xi)\}) + mV^m(\xi) + \xi U^m,$$

and so,

$$w_0^m(\mathcal{H}_i(\mathcal{I}_1 \times \dots \times \mathcal{I}_n)) \leqslant \kappa(\max_{i=1,\dots,n}\{w_0^m(\mathcal{I}_i)\}).$$

Now, by applying Theorem 8, the assertion is obtained. □

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